

DESIGN AND CALIBRATION OF A PORTABLE THREE COMPONENT SEISMOMETER

BY
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ERRATA

For TF 518:

Page 11, middle of page should read $\log \frac{y_1}{y_2} = \frac{T r}{2 M}$

Page 15, equation (8) should be,

$$f = \frac{F}{2} \cos (w_1 - w_2)t - \frac{F}{2} \cos (w_1 + w_2)t$$

Page 17, equation 19 should be $A = \frac{F}{4 (\pi)^2 f^2 M}$

For TP 556:

Page 13, Figure 9, diagrams for 49 and 63 cycles per second are upside down and interchanged.

Page 30, line 6, X sign should be omitted between words "radians" and "angular."

Page 31, equation (6), in next to last term "cos t" should read "cos wt"

Page 33, equation (10), last term of denominator should have + sign instead of —.

Page 33, equation (12), should have last two terms omitted, reading, $x_2 = C \sin (wt + \delta)$

Page 33, equation (15), the sign in the parenthesis should be +, making the parenthesis read $(w \cos \delta + a \sin \delta)$

Page 36, third line from the bottom of the page should have a + sign following the radical, $\sqrt{\frac{K}{K + 4w^2 LM}} +$

Page 40, Table 2, second value for $\sin \delta$ should be .0926, and third value for $w^2 LM + K$ should be 1.0×10^5

U. S. DEPARTMENT OF COMMERCE

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Technical Paper 518

CONSTRUCTION OF MASTER MECHANICAL OSCILLATOR FOR TESTING SEISMIC RECORDERS AND OTHER ALLIED APPARATUS

BY

F. W. LEE and G. A. IRLAND



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II



CONSTRUCTION OF MASTER MECHANICAL OSCILLATOR FOR TESTING SEISMIC RECORDERS AND OTHER ALLIED APPARATUS¹

By F. W. LEE² and G. A. IRLAND³

INTRODUCTION

Owing to the increasing importance of the study of mechanical vibrations from the viewpoint of seismic exploration as well as from the limit of strength a structure may offer to blasting operations, the Bureau of Mines has begun an investigation in this field. That harm to buildings may arise from blasting operations or from other sources of violent vibrational disturbances, there can be no doubt. However, just how great these vibrations may become before they are harmful is still a much debated question. Furthermore, it is a well-known fact that all buildings are constantly in a state of tremor or vibration; and just where the dividing line occurs which, if passed, will permanently damage the structures, is the important question of the moment.

Seismograms also, before they can be used to contribute real information concerning geophysics and the nature of vibrations, should be expressed in some absolute system of measurement. Records of different instruments at different stations should be directly comparable. It is therefore necessary to recheck or calibrate at various intervals these instruments to ascertain if they have retained their accuracy of calibration. It is also desirable, according to Walker,⁴ to examine the record of each instrument and compare it to the fundamental equations of motion inherent in its constructional design.

ACKNOWLEDGMENTS

The writers wish to express their appreciation for the cooperation in design and construction extended by R. H. Baker, J. A. Rowe, and H. A. Dent, of the Pittsburgh Experiment Station, Bureau of Mines, and to G. W. Cooke for assisting in the adaptation of the displacement recorder.

SPECIFICATIONS

It is therefore opportune to have available a suitable master oscillator or vibrator with which to calibrate the field apparatus. This master vibrator should—

1. Allow a direct measurement of displacement in a static or at rest position, using some recognized standard of comparison as, for example, a micrometric screw.

¹ Work on manuscript completed November, 1931.

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⁴ Walker, G. W., *Modern Seismology*: Longmans, Green & Co., New York.

2. Allow direct measurement of displacement in a vibrating or nonstatic condition.
3. Provide means for direct translation of the vibration into a corresponding change of current necessary to operate an oscillographic element.
4. Establish a permanent record of comparison between the seismometer on the vibrating table and the table itself.
5. Be capable of vibrating at any frequency and amplitude within very wide limits of frequency while supporting various seismometers.
6. Conserve the same percentage of accuracy of measurement at all amplitudes of vibration.
7. Have sufficient freedom of motion to respond to any predetermined transient vibration.

This type of apparatus distinguishes itself from what are commonly called shaking tables or platforms, referred to by Rogers,⁵ in that the flexibility of operation and precision are of another order. Also transients or vibrations of a continuously changing character may be produced at will with an apparatus of this kind. The usual shaking table is mechanically connected directly to a motor, thereby restricting its freedom of motion to narrow limits. It introduces friction of joints and other mechanical inaccuracies difficult to overcome and also to correct in the resultant observations. Purity of control is essential in apparatus of this nature.

SCALE OF MEASUREMENT

The convenient unit used for seismic recording is one one-thousandth millimeter or micron. It is approximately one-quarter of one ten-thousandth inch. In this work displacement alone is considered, not the velocity or acceleration of displacement, the reason being that a more direct physical relation exists between stress and strain than between stress and the velocity or acceleration of the strain.

This displacement is measured from the position of rest and is called the amplitude of vibration. From this definition it is clear that if the vibration is of a simple harmonic character the displacement will have the same value for both the positive and negative direction, and the total displacement is therefore twice the amplitude of vibration. This definition has the advantage that it avoids the introduction of a conversion factor and uses the same system of nomenclature common in the theory of sound, electricity, and optics.

Because of the complicated relations which arise in work of this nature it is very desirable to describe these conditions in the simplest way possible. It must be remembered that the damage which may be caused by vibrations having their origin in blasts or explosions not only causes the structure to tremble as a whole, but also creates relative displacements in the structural elements of the same. Under severe conditions these complicated displacements may be sufficiently great to rupture some material, resulting in what may be termed internal shattering. This phenomenon is analogous in action to the application of supersonic vibrations to a block of ice. Since here the wave length of vibration is extremely short the internal stresses on the crystal cleavage planes are sufficiently great to destroy their cohesion. As a result, after the vibrations have been applied the block of ice will have lost its strength and crumble. So it is also, with the passage of a very high-intensity high-frequency

⁵ Rogers, F. J., *The California Earthquake of Apr. 18, 1906: Report of the State Earthquake Investigation Commission*, vol. 1, pt. 2, pp. 326-335.

wave through a building, the individual portions of the building may become displaced beyond their limits of adhesion or cohesion. In the immediate vicinity of a blast this action may be desirable in shattering the rock, but its action should not extend too far beyond the blasting zone.

CONSTRUCTION

The main frame of the vibrator, Figure 1, consists of three similar iron castings, *a*, *b*, and *c*, each weighing about 200 pounds. Of these, *a* forms the top, *b* the bottom, and *c* the adjustable platform of the system. Four steel columns 2 inches in diameter, with an over-all length of 5 feet 8 inches, act as supports and guides to the above operating platforms. The columns are drawn down tight against a taper by suitable nuts *d*, thus assuring the required rigidity of the assembly. Table *c* also augments this rigidity of construction. For all practical purposes this table may be considered the working surface from which the relative degrees of motion are measured. This platform is also provided with special splines *e* and suitable clamping screws *f* for holding them in a level position. Very accurate leveling is necessary to constrain the vibration into only one invariant axis.

Upon this table, *c*, are placed three micrometer measuring screws *g*, two of which are seen in Figure 1. These are so placed upon table *c* that they can be made to bear against the bottom of the movable or vibrating table *h*. Special hardened-steel contact anvils are set into the lower face of this table to serve as micrometer contacts.

The table *h*, upon which the instruments undergoing calibration are placed, is made of aluminum five-eighths inch thick and is also heavily ribbed to avoid separate vibrations in its members. It is supported by four steel columns *i* to the top *j*, which is another rigid aluminum casting. Both *h* and *j* are made as light as possible by perforating them with 2-inch holes without sacrificing rigidity.

To the top of *j* is fastened a spring, *k*, which may be lowered by a hand wheel on the top of casting *a*, not shown in the figure. When the proper elevation for operation has been reached its position is further assured in being solidly clamped by hand wheel *l*.

The actuating or driving coil *m* is attached by two aluminum plates *n* to the bottom of vibrating platform *h*. These plates are also held rigid by stay bolts placed at proper intervals between the two plates. The maximum range of motion of this driving coil is five-sixteenths inch.

To restrict the motion of the table further to a purely vertical direction additional precautions were taken by applying flat steel restraining bands as ties between the table and the steel columns. These ties were fastened at one end by especially constructed studs *r* placed at the top and bottom of the movable table. The other end was clamped to a sliding collar on the steel column.

The rest of the driving mechanism was securely bolted to the bottom of casting *b*. An electrical panel, *p*, was provided for the 14 coil terminals incorporated in this portion of the apparatus.

PRINCIPLE OF OPERATION

The forces for moving this table are produced by a driving coil, *m*. A portion of this coil is in a magnetic circuit, and the force is proportional to the product of the current and the magnetic field strengths through which the current flows. To intensify this magnetic field

the operating air gap (fig. 2) is made as short as possible. Figure 2 also shows the form of magnetic yoke *b* made of laminated transformer iron. The magnetizing coils for this yoke are shown in *c* of this figure and the compensating coils by *d*.

To avoid any electrical interreaction from the moving coil *m* and the magnetizing coils *c* the number of turns in the two compensating coils *d* together are just equal to the number of turns in the driving coil *m*. The same current is passed through these compensating coils as is applied to the moving coil *m*, but in a reversed direction. By this arrangement the two magnetomotive forces always balance each other, and here no magnetism is introduced into the yoke by the current in the driving coils. Symmetry of magnetic as well as electric circuits constitutes the basis of good design.

A detail drawing of the construction of the driving mechanism is shown in Figure 4. The constants of design, at the left of the drawing, are so chosen that either alternating current or direct current may be applied to the driving coil as well as to the magnetizing coils. The driving or vibrating coil consists of No. 14 B. and S., carrying 16 amperes without difficulty. The magnetizing coils are composed of No. 10 wire and will carry 32 amperes without undue heating. For momentary loads these values may easily be doubled.

Connections are made so that the current in the upper and lower magnetizing coils oppose each other, thereby forcing the magnetism through the air gap.

The details of the construction of the vibrating coil are shown at the right of this drawing. Red fiber forms the main support, except in the air gap, where only the conductors of the coil are present.

ELECTRICAL MEASUREMENT OF VIBRATIONAL DISPLACEMENT

An electrical measuring system is used to check the optical system. It consists of a condenser plate 2 inches in diameter, mounted on the moving table, between two similar plates attached to operating platform *c*, Figure 1. As the mid plate of this condenser moves in a vertical direction it approaches one outer plate and recedes from the other. This measurement system with its micrometer adjusting screws is shown in Figure 3. The variations of capacity are converted into change of current by an oscillator recorder system. The current is photographically recorded by an oscillograph. In this arrangement the amplitude of vibration of the oscillograph may be measured upon a calibrated ground-glass screen and may be compared directly to the optical measuring system before a photograph is recorded. There is considerable advantage in such an arrangement because in this way the uniformity of deflection at different static positions of the table and at various amplitudes of vibration may be easily determined and also adjusted if necessary.

Transients of various kinds may be impressed upon the table by sudden changes of current in the magnetizing coils or in the driving coils. The character of these changes may be modified at will by the constants of the respective electric circuits. By placing a seismic recorder upon this table direct comparison can be made of the actual and recorded vibration.

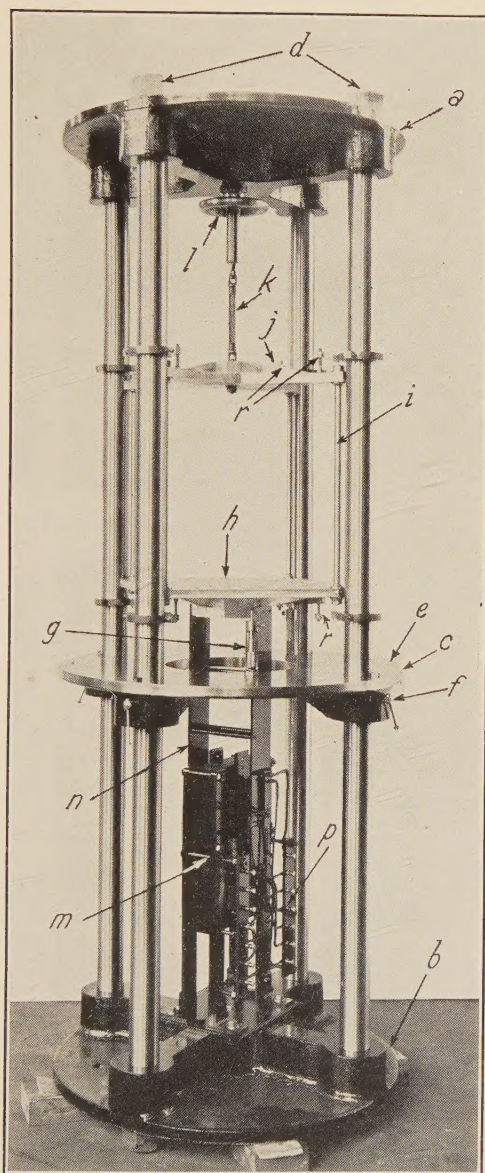


FIGURE 1.—Assembly of master vibrator

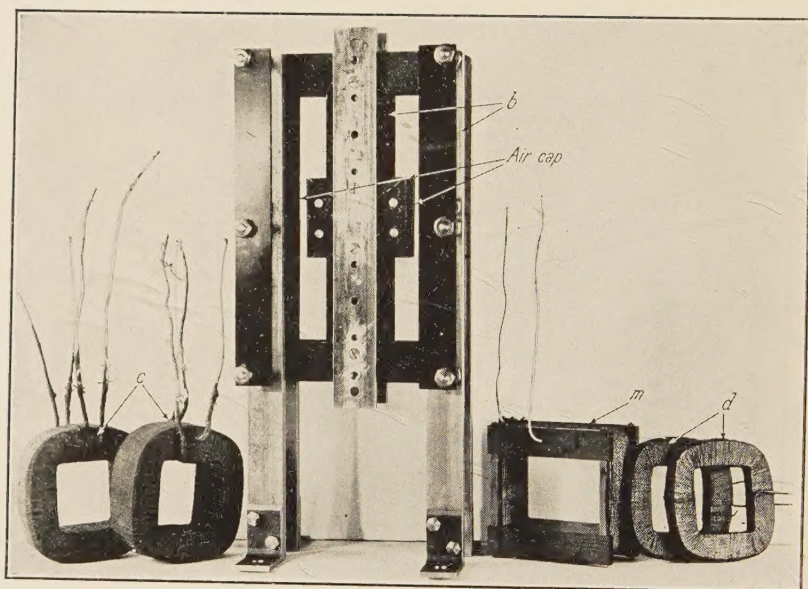


FIGURE 2.—Electrical driving mechanism of vibrator

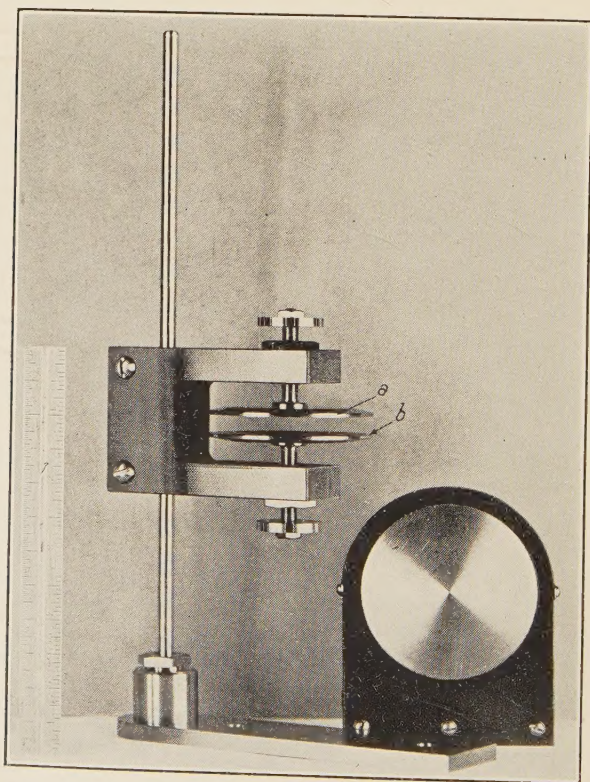


FIGURE 3.—Electrical condenser system for measuring amplitude of vibration

OSCILLATOR RECORDER

For the electrical measurement of displacement a change of electrical capacity is used as the actuating principle. The separation or approach of two surfaces lends itself to simple measurements of distance which is directly proportional to this capacity. Although changes of capacity have been used before by P. Duckert,⁶ H. Riegger,⁷ F. Trendelenburg, and others for seismic registration, their application depended upon a change of frequency usually functioning upon the straight portion of the electrical resonance curve.

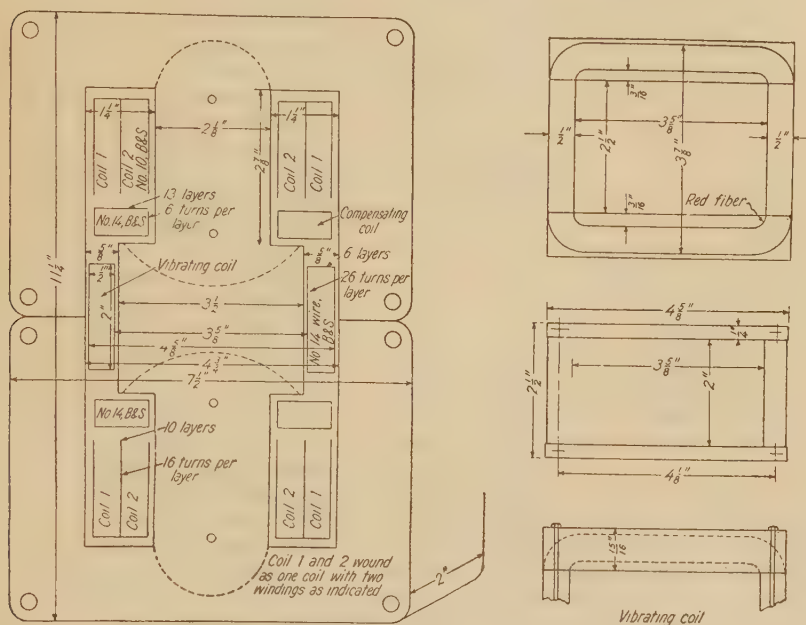


FIGURE 4.—Detail drawing of driving mechanism

Improvements in this system were suggested by Schaeffer in United States Patent 1717630 and were independently discovered and also used by G. W. Cooke. The principle relied upon is the change of synchronizing current that occurs in a coupling circuit between two independently oscillating electrical systems as one or the other system is modified by an accelerating or retarding influence. In this case the accelerating or retarding influence is due to the capacity changes of the system. Here the coupling current is directly proportional to the vector difference in voltage between the two systems. Even for differences in phase of 60° between the two main oscillating systems the difference in voltage is directly proportional to the angular displacement within an error of 2.2 per cent. Under this extreme condition the magnitude of this difference voltage would have the same value as the two equal synchronizing voltages. The theoretical curve would be a straight line, as shown in *a* of Figure 5. Actually the curve obtained

⁶ P. Duckert, *Ztschr. Instrumentenkunde*, vol. 46, 1926, p. 71.

⁷ H. Riegger, *Wiss. Veröffentl. a. d. Siemens-Korizern*, vol. 3, 1924, p. 67.

from the apparatus is shown at *b* of Figure 5. The relation between deflection and displacement is linear for the operating portion of this curve. It would be beyond the scope of this paper to give a complete mathematical analysis of a system working upon the above relations.

The electrical arrangement consists (see fig. 6) of the movable condenser plate 2, constituting the electrical capacities c_1 and c_2 between fixed plates 1 and 3. Thus

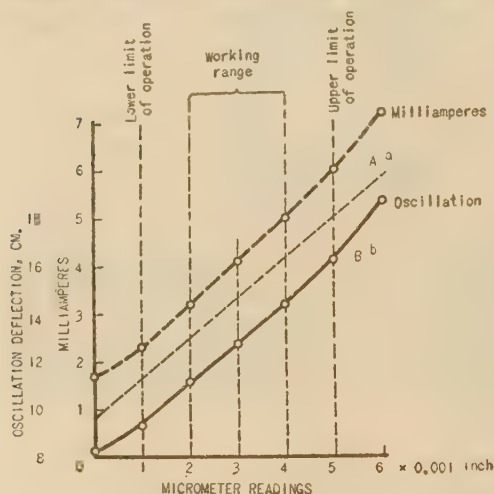


FIGURE 5.—Calibration curves of oscillator recorder

plate 2, in moving up, increases capacity c_1 and decreases capacity c_2 . Inductances l_1 and l_2 , with capacities c_1 and c_2 , form the two high-frequency oscillating systems. The small trimming condensers 4 and 5 adjust these frequencies into the best operating range. Oscillating systems c_1 and l_1 , also c_2 and l_2 , are held in a cyclical state with vacuum tubes 6 and 7 operating with the conventional "magnetic feed back" coils 8 and 9. A grid leak and condenser, 10 and 11, are used in the grid circuit to supply negative potential to the grids. Two coils, 12 and 13, are connected to the vibrating systems and serve to electrically couple them together. Milliammeter 14, rectifying bulb 15, and oscillographic circuit 16 are also inserted in this circuit. There is sufficient capacity, 17, in system to allow the synchronizing current to by-pass this high-impedance circuit of rectifying tube 15, and the milliammeter. Since tube resistance 15 is very high the circuits to the oscillograph will not materially affect the functioning of the system.

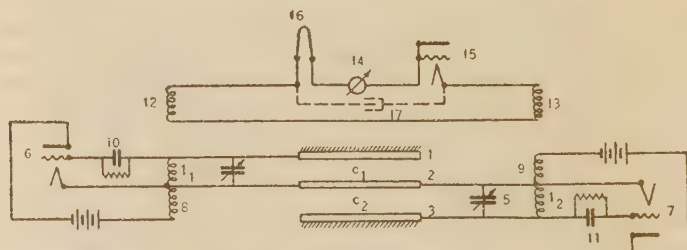


FIGURE 6.—Diagrammatic sketch of oscillator recorder

This arrangement avoids the use of very powerful amplifying systems which are difficult to calibrate and also maintain in calibration. A micrometer adjustment on condenser plate c_2 allows direct calibration, using constant increments of displacement. Both the reading of the milliammeter and the oscillograph are observed. When plate 2 is vibrating this frequency does not materially change the current in the oscillographic circuit because of its extremely high ohmic resist-

ance. A high-frequency shunt may be placed across the milliammeter at extremely high vibration frequencies. Furthermore, the impedance in the coupling circuit must be inductive if the system is to regulate for zero current.

Actually the circuit is somewhat modified in practical application. (See fig. 7.) Only one filament and plate battery are used; the latter is also shunted by a common condenser. The system is operated from a 90-volt plate battery and two dry cells for the filament. Figure 8 shows a front view, with the filament-control rheostat and the trimming condensers 4 and 5. Care is necessary in the length of leads to the condensers and should be made as short as possible. Figure 9 shows the arrangement of the material in the aluminum box.

This circuit arrangement is also well suited for microphone-transmitter construction and the operation of a capacity "pick-up" upon

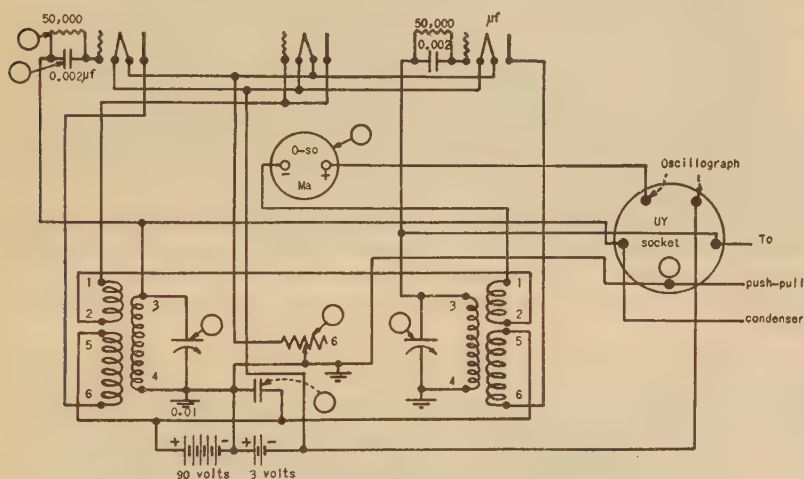


FIGURE 7.—Wiring diagram of oscillator recorder

graphophone records. Only the sidewise motion registers and the scratch at the bottom of the groove is eliminated.

MATERIALS FOR CONSTRUCTING OSCILLATOR RECORDER

The following materials are needed for constructing the oscillator recorder:

One aluminum box, No. 12 B and S, 5 by 9 by 6 inches.
Three UX spring sockets.
Two resistors, 50,000 ohms.
Four mounting clips.
Two coil assemblies.
Two bakelite washers.
Two brass rods, 2 inches by 1/4-inch diameter.
One-quarter pound No. 28 D. C. C. magnet wire.
One fixed condenser, 0.01 mfd.

One bracket.
Two variable condensers, 100 mmfds.
One UY socket.
One milliammeter, 0-10 range.
One rheostat, 6-ohm.
Two fixed condensers, 0.002 mfd.
Two condenser supports.
One 6-prong socket.
One 6-prong coil.
Two 231 R. C. A. tubes or equivalent.
One 230 R. C. A. tube or equivalent.

OPTICAL MEASURING SYSTEM

An optical measuring system is used to determine the amplitude of vibration. It consists of a small concave mirror mounted on a shaft in the customary "optical lever" arrangement. A light copper wire is attached to the movable table and passes around the shaft of the mirror axis. The other end may either be held taut by a spring or be again fastened in a taut condition to the movable table. Light passes through a slit, and the image of the slit is focused by the mirror upon a millimeter scale. In this way the movement of the table is greatly magnified, and magnitudes of vibration less than 1 micron may easily be measured. By using various shaft diameters the range of measurement may be further amplified.

The exact displacement of the optical system is first measured from the static condition by changing the elevation by the micrometer screws. In an oscillatory condition the same amplitudes are again observed by the width of the band of light on the scale.

METHOD OF MANIPULATION

The seismometer to be examined is placed upon the table, and the vibration platform is gradually raised with the hand wheel at the top of the apparatus. The exact amount of raising is read directly from the optical scale.

The electrical registration is now adjusted by separating or bringing the two plates *a* and *b* of Figure 4 closer together, depending upon the sensitivity desired. For large vibrations they are farther apart. It is necessary to know in advance the range of amplitude of operation desired.

The relation between the amplitude of vibration as recorded by the optical system connected to the table and the deflection upon the ground glass in the oscillograph must be proportional. The oscillograph owes its vibration to the change of capacity in the recorder system.

After this is done it is generally desirable to apply the same amplitude of vibration through a very wide range of frequencies and to compare the recorded amplitude in magnitude and phase with that of the table.

Since in this system both may be recorded simultaneously upon the same paper film it is relatively simple to measure the relative accuracy of amplitudes at different frequencies.

From general equation (19)—see appendix—when operating at frequencies removed from the resonant frequency the relation between the amplitude of motion and applied force was given by the relation:

$$A = \frac{F}{M\omega^2} = \frac{K_2 I_{dc} I_{ac}}{M\omega^2} \quad (20)$$

$$A = \frac{K}{M} \cdot \frac{I_{dc}}{\omega^2} \cdot \frac{I_{ac}}{\omega^2}$$

The equation clearly shows if $\frac{I_{ac}}{\omega^2} = \text{constant} = K_a$.

or

$$I_{ac} = K_A \omega^2$$

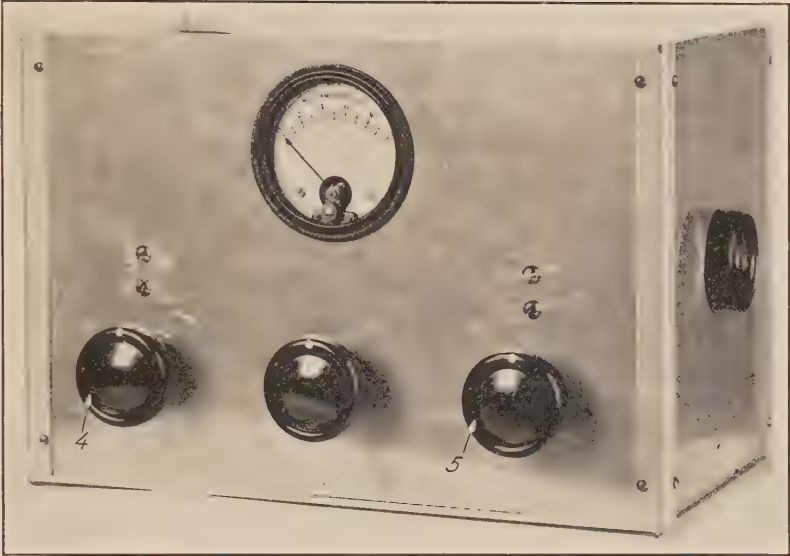


FIGURE 8.—Assembly of oscillator recorder

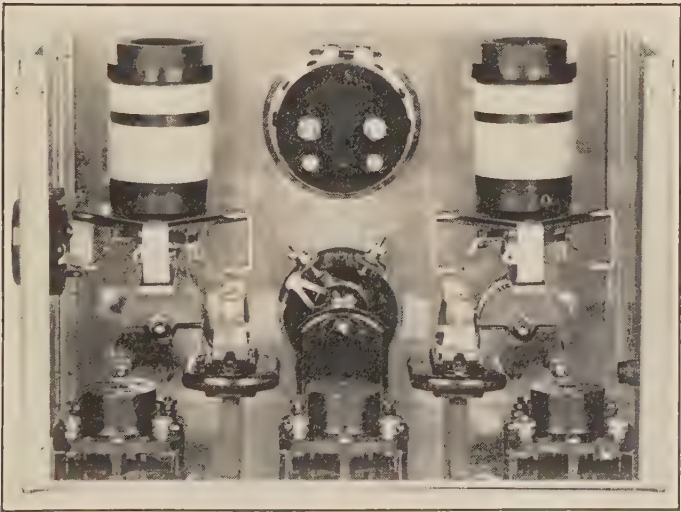


FIGURE 9.—Detail view of assembly of oscillator recorder

then the amplitude of vibration will be very nearly constant. By altering the excitation of the alternator it is relatively simple to adjust this current to the proper value demanded by this equation.

From equation (20) it is seen that amplitude of motion may also be controlled by changing the magnetizing current. Owing to the change of magnetic saturation in the iron core this procedure introduces an element of error which may or may not be objectionable.

MEASUREMENT OF CONSTANTS OF VIBRATOR

There are many ways of measuring the constants that characterize the motion of an instrument of this nature. That the constants determined under static conditions do not agree with those taken under dynamic conditions is to be expected because some of the so-called constants are not strictly constant.

It is safe to assume that the mass under vibration does not change with the frequency. On the other hand, the uniform relation between the force on the spring and the extension involves the frequency of vibration. It is quite well known that if an added constant force is applied to a spring considerable time will be required before it arrives at its final state of elongation or compression. If the cyclical changes occur very rapidly the spring will not have had sufficient time to reach this position, with the result that it will appear stiffer than it really is. The resistance to motion is also not strictly proportional to the velocity of motion nor the same for every weight on the platform or extension of the spring. The same may also be said of the incorporated electrical system; however, the factors are more restricted, involving primarily only the saturation of the iron core. In other words, the percentage increase of the magnetic field of the driving coil is not strictly proportional to the magnetizing current, especially at very low and very high magnetic intensities or flux densities.

MEASUREMENT OF CONSTANTS

Determination of force per unit product of alternating and direct currents.

From data of Figure 10 the average ratio of

$$\frac{\nu^2}{\sqrt{2} I_{ac} I_{dc}} = \frac{37.9}{\sqrt{2}}$$

for constant amplitude of vibration and table empty.

Considering the relation

$$A = \frac{F}{M\omega^2} = \frac{K_2 I_{ac} I_{dc}}{M (2\pi)^2 \nu^2} \quad (20)$$

$$K_2 = A \cdot M \cdot (2\pi)^2 \cdot \frac{\nu^2}{I_{ac} I_{dc}}$$

The alternating current as measured by an ammeter must be multiplied by $\sqrt{2}$

$$= 0.001 \times 2.54 \times 39.5 \times \frac{37.9}{\sqrt{2}} \times 12,400$$

$$= 33,300 \text{ dynes per unit product of currents.}$$

This compares with $K=37,100$ dynes per unit product for the static condition (fig. 11). In other words, if the force is applied for a very long time, as in the static case, it will elongate the spring farther and appear to give a much higher force per unit product of the two currents. Also, the change of magnetic saturation tends to decrease the pull, of the alternating current magnetic effect, in the air gap.

Time scale.—A very sharp curve of time impulses is obtained by the "blocking" of the plate current in a high-frequency oscillator by letting the grid accumulate a high negative charge. After the oscillations cease the grid loses its charge, and the system begins once more to oscillate. By applying a known frequency, Figure 12a, of 59.2 cycles per second and counting the cycles for 10 time intervals, in this case 64, sufficiently accurate time determinations are possible.

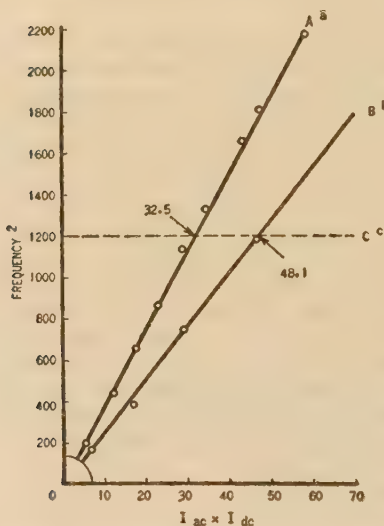


FIGURE 10.—Normal operating characteristic of vibrator

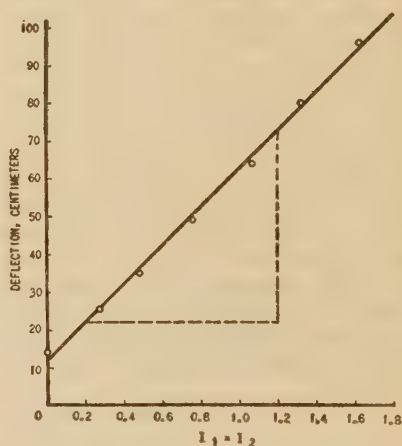


FIGURE 11.—Electrical direct-current characteristic of driving mechanism

Here one time division is equivalent to 0.108 second, and one-fiftieth inch is equivalent to 0.00815 second.

Determination of mass.—The mass may be determined by applying different frequencies to the oscillating system and plotting the square of this frequency against the product of $I_{ac} \times I_{dc}$. According to equation (20), this should be a straight line. The observed results are shown in Figure 10. Curve *a* represents the vibration with the table empty and curve *b* with the table loaded, with a 6-kilogram mass adjusted to the same amplitude of vibration.

By selecting a common frequency line, *c*, Figure 10, remembering $A = \frac{F}{M\omega^2}$, under constant amplitude reduces to

for

a

$$I_{dc} \times I_{ac} = K \cdot M \cdot \omega^2$$

$$32.5 = K \cdot \omega^2 \cdot M$$

b

$$48.1 = K \cdot \omega^2 \cdot (M + 6).$$

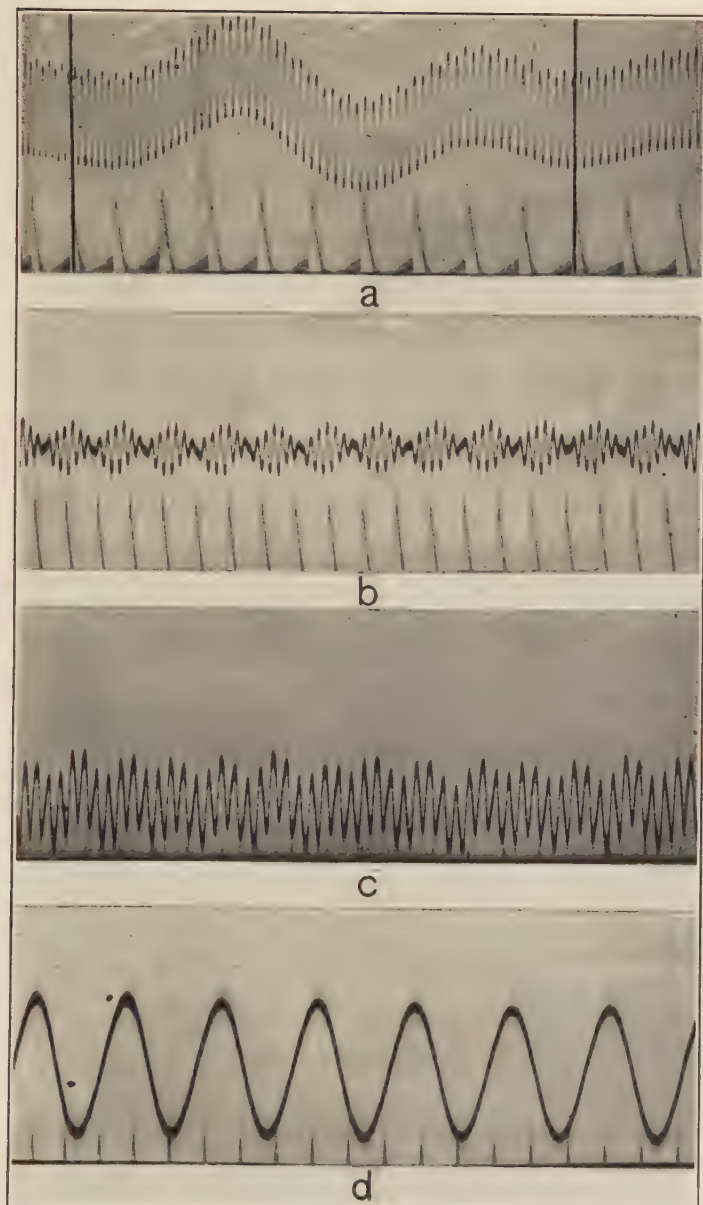


FIGURE 12.—*a*, Method of determining time scale; *b*, *c*, *d*, mechanical beat phenomena

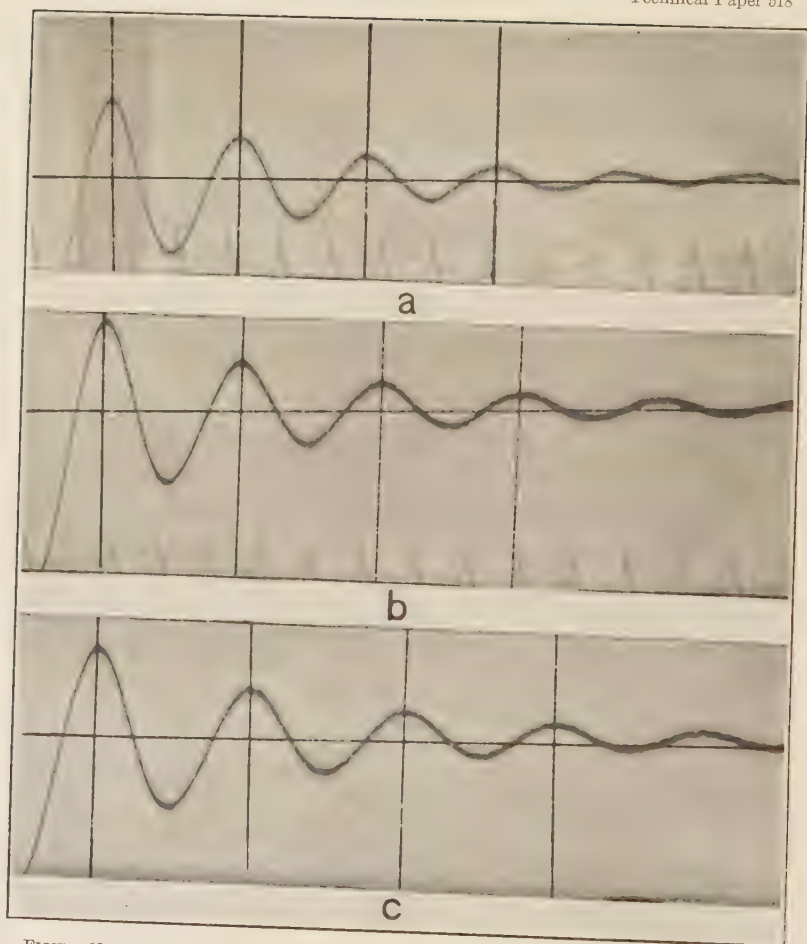


FIGURE 13.—Normal decay of vibrations of master vibrator with various masses on the table

Dividing and solving for the mass M ,

$$M = 12.4 \text{ kilograms.}$$

Unfortunately, the machine was assembled before the separate parts composing the table were weighed, and this alternative was used to determine its mass.

Determination of r .—Resistance to motion or determination of constant r can best be obtained from the normal decay of amplitude curve. The equation of the envelope of curve, Figure 13, is of the form

$$y = Ae^{-\frac{r}{2M}t}$$

If the points of contact are chosen, the intervals are the time of a period, or cycle. Let the time, $t = t_1$ and t_2 for these successive intervals; then

$$y_1 = Ae^{-\frac{r}{2M}t_1} \text{ or } \log y_1 = -\frac{r}{2M}t_1 + \log A$$

$$y_2 = Ae^{-\frac{r}{2M}t_2} \text{ or } \log y_2 = -\frac{r}{2M}t_2 + \log A$$

and

$$\log y_1 - \log y_2 = -\frac{r}{2M}t_1 + \frac{r}{2M}t_2 = \frac{r}{2M}[t_2 - t_1] = \frac{rT}{2M}$$

$$\log \frac{y_1}{y_2} = \frac{Tr}{2M} = \frac{r}{2M\nu}$$

Where T = period, $\nu = \frac{1}{T}$ = frequency = 3.71 cycles per second.

From Table 1, $\log \frac{y_1}{y_2} = \log \frac{1}{0.537} = 0.623$

$$r = 2M \cdot \nu \cdot (0.618) = 2 \times 12,400 \times 3.71 \times 0.618 = 57,300 \text{ dynes per centimeter per second.}$$

The above condition is for the table empty. With the table loaded, Figure 13, c , with 5 kilograms, $r = 66,700$ dynes per centimeter per second.

TABLE 1.—From Figure 13

Time		Ordinate	
0		20.5	} 0.536
1 $\frac{1}{2}$ T	Ratios for negative values {	19.0	
T		11.0	
3 $\frac{1}{2}$ T		10.	} .591
$2 T$	{ .5	(6.50)	
5 $\frac{1}{2}$ T		(5.0)	
3 T		3.5	} .531

$$\text{Mean ratio} = \sqrt[5]{0.536 \times 0.527 \times 0.591 \times 0.5 \times 0.531} = 0.537.$$

$$\text{Logarithmic decrement} = \log_e \frac{1}{0.537} = 0.623.$$

TABULATION OF CONSTANTS OF MASTER VIBRATOR

Driving coil:		
Direct-current resistance of driving coil.....ohm---		0.526
Direct-current resistance of lower compensating coil.....do----		.184
Direct-current resistance of upper compensating coil.....do----		.190
Total direct-current resistance.....do----		.900
Total alternating-current resistance (50 cycles).....do----		.976
Inductance of driving coil and compensating coils.....henry---		.00566
Magnetizing coils:		
Coil No. 1—		
Direct-current resistance, No. 1 winding.....ohm---		.072
Direct-current resistance, No. 2 winding.....do----		.091
Coil No. 2—		
Direct-current resistance, No. 1 winding.....do----		.072
Direct-current resistance, No. 2 winding.....do----		.091
Total direct-current resistance (in series).....do----		.326
Total alternating-current effective resistance.....do----		*1.350
Alternating-current effective inductance.....henry---		.043
M =mass of vibrating table (empty).....grams---		12,400
K_1 =dynes per unit stretch of spring.....centimeters---		3,180,000
r =dynes per unit velocity—		
Impedance (table empty).....centimeters per second---		57,300
Impedance (table loaded).....do----		66,700
K_2 =dynes per unit product of current in driving and magnetizing coil—		
Direct current.....		37,100
Alternating current.....		33,500
from design constants (fig. 3).....		38,000
Maximum amplitude of vibration.....centimeter---		.200
Logarithmic decrement.....do----		.623
Maximum sensitivity of oscillographic recorder—		
Micron.....		.2075
Centimeter.....		.0002075

DISCUSSION AND RESULTS

The results of the investigation clearly disclose that mechanical vibrations are susceptible to quite accurate measurements. Vibrational displacements may be measured to within an accuracy of from 1 to 5 per cent without ultrarefinements. By careful construction of apparatus it is possible to compute the performance of vibration systems from the inherent constants of the system much in the same way as in alternating current circuits.

Figure 13 shows the effect of increasing the mass of a vibrational system when left to itself. The first curve (*a*) is for the table empty, the second (*b*) loaded with 2.5 kilograms, and the third (*c*) with 5 kilograms. Similarly shaped curves represent the decay of an electric current in a circuit having resistance, capacity, and inductance as the inductance is increased. The vertical lines clearly show the increase of the natural period of the system.

When two different frequencies of current are applied to the mechanism the beat phenomenon occurs as is shown in Figure 12, *b*. The frequency of the driving coil was 54 cycles per second and of the magnetizing coil 2.5 cycles per second. The sum and difference are the result; that is, 56.5 cycles and 51.5 cycles per second or a beat frequency of 5 cycles per second. If the two frequencies are 20.5 and 53.9 cycles, Figure 12, *c*, equivalent to 84.4 and 33.4 cycles,

* Includes effect of iron.

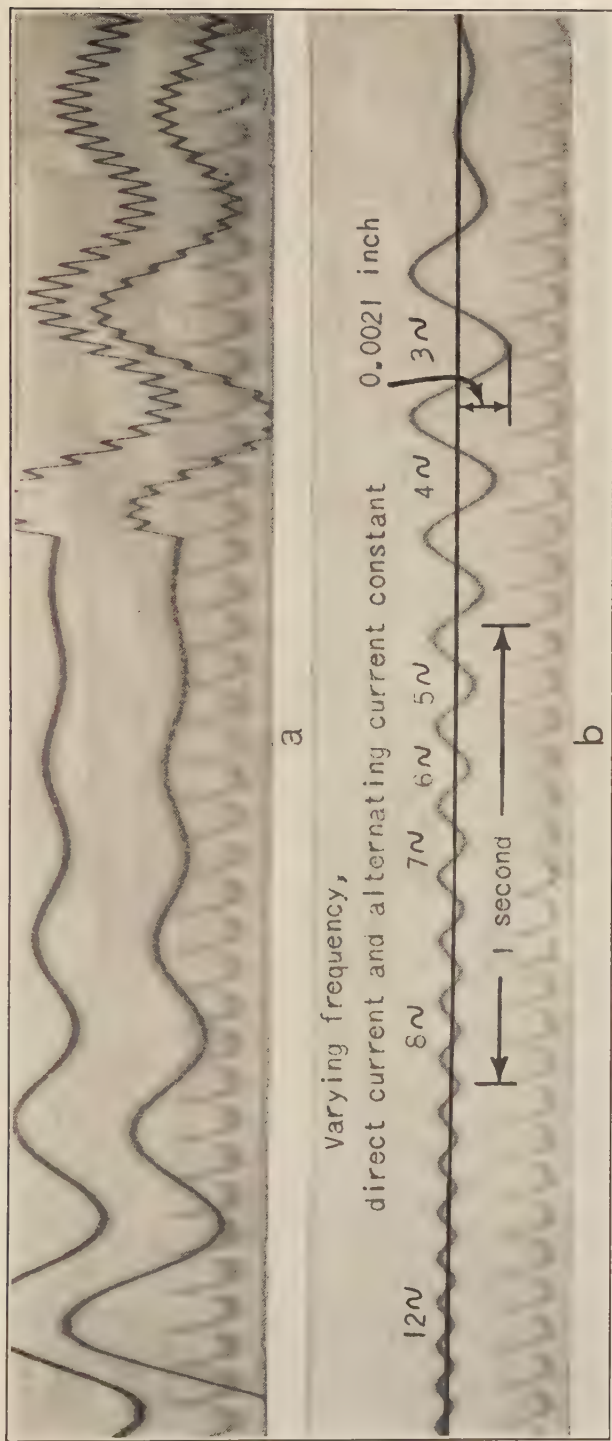


FIGURE 14. *a*, Comparison of seismometer transient vibration to master vibrator; *b*, frequency characteristics of master vibrator

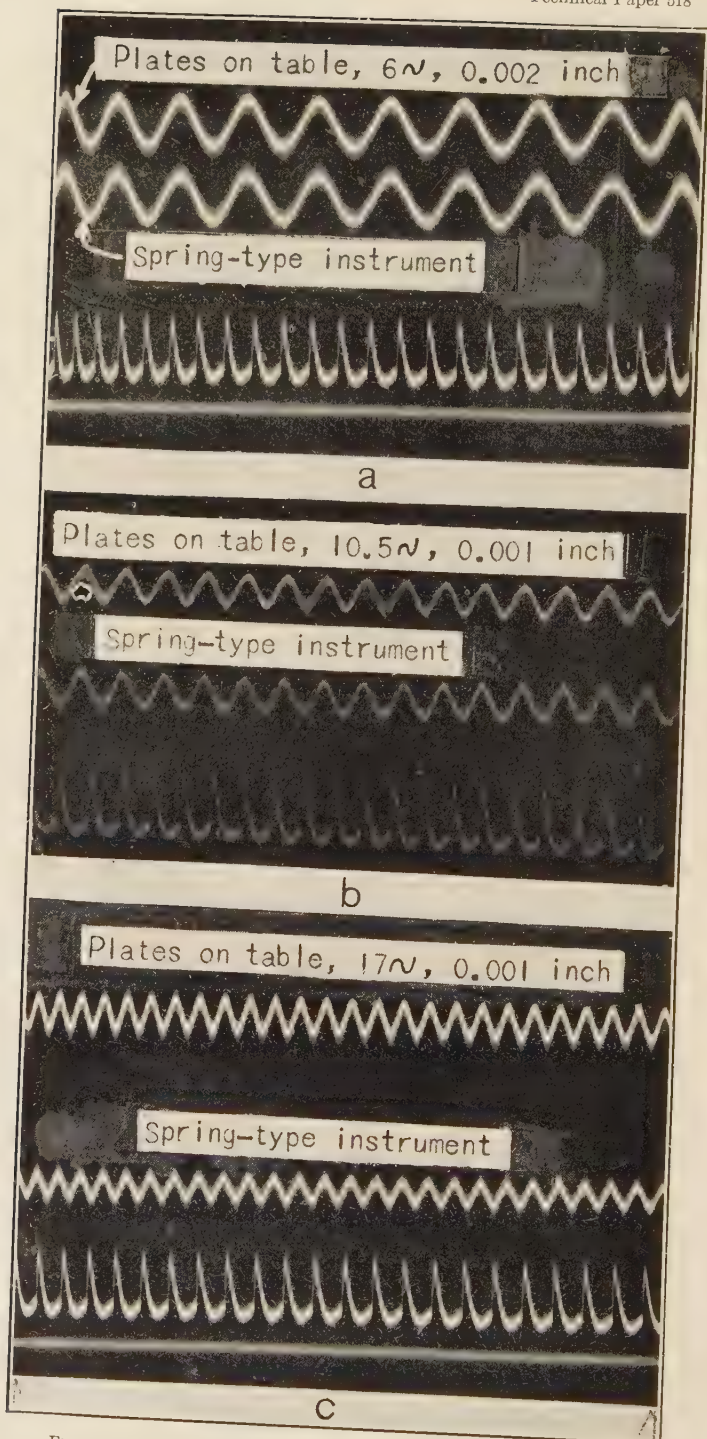


FIGURE 15.—Method of comparing seismometer with table vibrations

the phenomenon of the beat frequency of 51 cycles begins to become more imperceptible. This must be so, since the impedance to motion for the same force at a high frequency becomes very great relative to a low frequency. This is further illustrated in Figure 12, *d*, where the two frequencies are 54.6 and 50.7 cycles. The resultant effect is just one frequency, the difference frequency of 3.9 cycles; the amplitude of the high frequency of 105.3 cycles is relatively insignificant. That this is so is brought out by Figure 14, *b*, which shows the instrument to be especially responsive in the vicinity of 4 cycles and gradually losing responsiveness with higher frequencies.

How a seismometer may be calibrated at various frequencies and amplitudes is shown in Figure 15. In part *a*, 6 cycles per second with 0.002-inch total displacement was used. In part *b* the frequency was increased to 10.5 cycles and the amplitude reduced by one-half. Part *c* shows a further increase in frequency to 17 cycles at the same amplitude. The spring-type seismometer as here used is shown to be selective in this frequency range, falling off in sensitivity at the higher frequencies.

In Figure 14, *a*, two curves are again shown, the upper one that of the table and the lower that of the instrument. A simple transient is indicated in the first part of this curve, followed by a more complicated one in the latter part of the picture. True fidelity is shown in the first part for slow vibrations with a very marked decrease of sensitivity to the higher frequencies in the second part of the film.

Mechanical systems free from linkages having vibrational rigidity are especially well suited for mathematical analysis of this nature. That building structures, units, etc., should fall into this class is purely coincidence.

It is suggested that the c. g. s. system, on account of its intrinsic simplicity, be adopted in describing these relations and that the mechanical definitions which arise have a nomenclature similar to that used at present in the electrical science.

From the relation

$$A = \frac{\frac{F}{\omega}}{\frac{K}{\omega} - M\omega + jr} = \frac{\text{Force per unit angular velocity}}{\frac{K}{\omega} - M\omega + jr}$$

the system may be reduced to one of linear fractional transformation of the straight line $\frac{K}{\omega} - M\omega + jr$ into a circle having a diameter $\frac{1}{r}$, allowing a simple graphical visualization of the amplitudes and phase relationship, the variable parameter being the angular velocity or frequency when the factor $\frac{F}{\omega}$ is held constant.

APPENDIX.—ELEMENTARY MATHEMATICAL RELATIONS CONTROLLING VIBRATION OF TABLE

By restricting the vibration to a vertical direction a great simplicity of motion exists, and the mathematical analysis is greatly simplified. This is spoken of as one degree of freedom of motion. Since the actuating force depends upon the product of two currents, each independent of the other, two further degrees of freedom are intro-

duced. To give a full discussion of the complete analysis of motion would extend beyond the scope of this paper; and the following simplified exposition will generally suffice for most purposes. See, however, F. Kiebitz.⁹

Under the above construction, applying the well-known equation of a free body to the table, the following relation exists:

$$\text{Mass} \times \text{acceleration} = \text{applied force} - \text{retarding forces} \quad (1)$$

Let M = mass of moving body, grams,

s = displacement, centimeters,

f = applied force expressed as a function of time, dynes,

r = resistance to motion due to damping, which for slow speeds is proportional to the velocity of displacement

$\left(\frac{ds}{dt}\right)$, dynes per centimeter per second. (See Jacobsen¹⁰ and Den Hartog.)¹¹

K_1 = spring constant, dynes per centimeter stretch of spring.

Inserting these values in equation (1),

$$M \frac{d^2s}{dt^2} = f - r \frac{ds}{dt} - K_1 s, \quad (2)$$

generally written as

$$M \frac{d^2s}{dt^2} + r \frac{ds}{dt} + K_1 s = f. \quad (3)$$

In order to examine further

Let $V_1 = \phi(t)$ express the time variation of the current in the driving coil, and $V_2 = \psi(t)$ express the time variation of the current in the exciting coil.

By neglecting the nonlinear effect of saturation in the iron

$$f = K_2 V_1 V_2 = K_2 \phi(t) \psi(t), \quad (4)$$

where K_2 is a geometrical constant of the mechanism.

If the current in the two coils is constant, then

$$\begin{aligned} \phi(t) &= I_1 \text{ and } \psi(t) = I_2, \text{ and} \\ &= K_2 I_1 I_2 = F. \end{aligned} \quad (5)$$

This represents, therefore, a constant force applied to the table. The table under this condition will take a definite displacement. (See fig. 11.) If now $\phi(t)$ or $\psi(t)$ be suddenly reduced to zero,

$$\phi(t) = 0 \text{ or } \psi(t) = 0.$$

Then $f = 0$ and equation (3) becomes

$$M \frac{d^2s}{dt^2} + r \frac{ds}{dt} + K_1 s = 0. \quad (6)$$

The solution will be the transient of motion, a decaying oscillation shown in Figure 13, reaching constant displacement very gradually.

⁹ Kiebitz, F., Ann. Phys., vol. 40, 1913, p. 133.

¹⁰ Jacobsen, L. S., Bull. Seismol. Soc. America, vol. 20, No. 3, pp. 196-223.

¹¹ Hartog, J. P. Den, London, Edinburgh, and Dublin Philosophical Magazine, 7th ser., vol. 9, May, 1930, p. 801.

If the current in the exciting coil is constant and an alternating current is applied to the driving coil

$$\psi(t) = I_2 \text{ and } \phi(t) = I \sin \omega t,$$

in which $\omega = 2\pi \times \text{frequency}$;

equation (3) becomes

$$M \frac{d^2 s}{dt^2} + r \frac{ds}{dt} + K_1 s = (I_2 K_2) I \sin \omega t = F \sin \omega t, \quad (7)$$

where $F = \text{constant amplitude factor}$.

Under a steady condition this is a harmonic vibration at the impressed or forced frequency. (See fig. 15.)

If two different frequencies are used, one in the driving coil and the other in the magnetizing coil,

$$\begin{aligned} \phi(t) &= I_1 \sin \omega_1 t \text{ and } \psi(t) = I_2 \sin \omega_2 t, \\ f &= K_2 I_1 I_2 \sin \omega_1 t \sin \omega_2 t, \\ &= F \sin \omega_1 t \sin \omega_2 t, \end{aligned}$$

where $F = K_2 I_1 I_2$ is the constant amplitude factor,

since $\sin \omega_1 t \sin \omega_2 t = \cos \frac{(\omega_1 t - \omega_2 t)}{2} - \cos \frac{(\omega_1 t + \omega_2 t)}{2}$.

$$f = \frac{F}{2} \cos (\omega_1 - \omega_2) t - \frac{F}{2} \cos (\omega_1 + \omega_2) t, \quad (8)$$

which states that the system oscillates at two frequencies equivalent to the sum and difference of the separate frequencies. See Figure 12b. This is the familiar beat phenomenon.

If the currents i_1 and i_2 are transients, and the respective constants of each circuit do not change,

$$\phi(t) = \frac{E e^{i \omega_1 t}}{Z(p) p = j \omega_2} - \sum_{n=1}^n \frac{E e^{p_n t}}{(j \omega_2 - p_n) \frac{\partial z}{\partial p} p = p_n}. \quad (9)$$

$\Psi(t) =$ a similar expression as (9) for the other electrical circuit. The first term of equation (9) is the final steady vibration, the second term the transient. The value of p_n is the frequency discriminant of the electrical system (see fig. 14, a). For further details see W. Deutsch.¹²

All of the above different force functions may be symbolized by

$$f = \sum_{n=0}^{n=\infty} F_n \Sigma e^{n \omega t} \quad (10)$$

where $F_n = \text{constant factor or force for the value of}$

$\omega = \omega_1 + j \omega_2$, a complex angular velocity composed of ω_1 hyperbolic radians per second and ω_2 circular radians per second and n is the harmonic in question under consideration.

If $n=0$ then $f=F$ of equation (5).

When $n=1$, then $f=F$ of equation (7).

When $n=r$ and also $n=s$, where r and s represent any two numbers, then $f=F$ of equation (8).

More complicated beats may be introduced by choosing more factors or values of n .

¹² Deutsch, W., Archiv für Electrotechnik, B. 6, 1918.

Usually the value of $\omega = j\omega_2$, in which the force is of a purely harmonic or repeating character.

Equation (3) may therefore be generalized as

$$M \frac{d^2 s}{dt^2} + r \frac{ds}{dt} + K_1 S = \sum_{n=0}^{n=\infty} F_n e^{j n \omega t} + 0. \quad (11)$$

While equation (11) may at first appearance look very formidable, actually it is very quickly solved, since from the theory of differential equations the solutions are all of an additive character, and it is only necessary to solve one case, the rest being all similar to this one.

For example,

$$\text{let } n=1 \text{ and } F = |F| e^{j \omega t},$$

which means that the reference time is chosen arbitrarily 0. Now equation (11) becomes

$$M \frac{d^2 s}{dt^2} + r \frac{ds}{dt} + K_1 S = |F| e^{j \omega t}. \quad (12)$$

The general form of the solution is

$$S = A \Sigma e^{j \omega t}.$$

Substituting this value in equation (12),

$$A = \frac{|F|}{+M\omega^2 + K_1 + r\omega} = \frac{|F|}{Z_1} = \text{amplitude of vibration.} \quad (13)$$

Where $M\omega^2 + K_1 + r\omega$ is the generalized impedance to displacement. If

$$\omega = \omega_1 + j\omega_2 = 0 + j\omega_2$$

that is, the applied force is purely harmonic in character, $j = \sqrt{-1}$,

$$A = \frac{|F|}{-M\omega_2^2 + K_1 + jr\omega_2} = \frac{\frac{|F|}{\omega_2}}{-M\omega_2 + \frac{K_1}{\omega_2} + jr} \quad (14)$$

The solution under steady condition is therefore

$$S = \frac{|F|}{-M\omega_2^2 + K_1 + jr\omega_2} e^{j\omega_2 t};$$

therefore also

$$= \frac{|F|}{\sqrt{(-M\omega_2^2 + K_1)^2 + (r\omega_2)^2}} \sin \left(\omega_2 t + \tan^{-1} \frac{r\omega_2}{K_1 - M\omega_2^2} \right), \quad (15)$$

as observed on oscillograph.

When

$$\begin{aligned} \omega_2 &= 0, \\ |A| &= \frac{|F|}{K_1}. \end{aligned}$$

From equation (5)

$$|A| = \frac{K_2 I_1 I_2}{K_1} \quad (16)$$

This states that direct current is applied to both coils. For frequencies of higher values, if r and K are small compared to $M\omega^2$,

$$A = \frac{|F|}{M\omega^2}. \quad (17)$$

This is the ideal equation of design for seismometers.

For frequencies of low value, $\frac{K}{2} > M\omega^2$, and negligible resistance to motion,

$$A = \frac{|F|}{K_1}. \quad (18)$$

This indicates a relative reversal of phase with reference to the applied force for very low frequencies.

The table, if operated at frequencies higher than 10 cycles a second, reacts as a pure inertia, and the resistance, as well as the elastic constants of the spring, becomes irrelevant.

Under this condition the amplitude of motion and the applied force have the following simple relation

$$A = \frac{F}{M\omega^2}, \quad (19)$$

or

$$A = \frac{F}{4\pi^2\nu^2 M}.$$

Since

$$\begin{aligned} \omega &= 2\pi\nu, \\ \nu &= \text{frequency in cycles per second.} \end{aligned}$$

From the standpoint of calibration this change of phase does not enter into the situation. However, in the functioning of seismic recording apparatus, this condition is very objectionable if present in the operating range of the instrument. Details concerning these features will be presented in a later paper.



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BUREAU OF MINES

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Technical Paper 556

A STUDY OF SOME SEISMOMETERS

BY

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A STUDY OF SOME SEISMOMETERS¹

By G. A. IRLAND²

SUMMARY

This paper discusses the advantages of seismometers that record displacement over instruments that measure velocity or acceleration. It shows that a portable seismometer using changes in capacitance between condenser plates to produce proportionate changes of current in an oscillograph has many good characteristics, chief of which are the ease and accuracy with which calibration may be made in the field after sensitivity has been adjusted as desired.

Several types of vertical-component instruments were built and tested on the mechanical oscillator. These are described and their characteristics discussed. Oscillograph records for transient as well as steady-state vibrations having frequencies of 2 to 70 cycles per second show curves of actual motion on the same record with those registered by the seismometer. Peculiarities at critical frequencies can thus be discovered and corrected. Finally, a vertical-component seismometer in which the change in pull of the supporting spring is compensated by change in radius arm at which the spring force acts was developed during the investigation. Performance of this instrument is shown to be excellent for a frequency range of 2.5 to 22 cycles per second. As earth vibrations due to blasting usually have a frequency of 10 to 15 cycles per second this instrument is well suited for investigating disturbances of this type. A horizontal-component seismometer having approximately the same design constants as the vertical-component instrument was also developed and is described.

The appendix gives a mathematical analysis of the motion of the center of oscillation of the seismometer arm and offers examples of computations by which performance of an instrument may be determined theoretically.

The importance of fidelity in recording transient effects is discussed; seismometers should be so designed that the initial value of the transient effect is negligible. Tests on the mechanical oscillator show good agreement between curves of actual motion and seismometer records for complicated transients.

Direct spring support for the arm of a vertical-component seismometer is shown to be unsatisfactory. The theory of geometrical compensation is developed, and computations are made showing application of the theory to actual design.

¹ Work on manuscript completed August 1932.

² One of the consulting engineers, U.S. Bureau of Mines, Bucknell University, Lewisburg, Pa.

INTRODUCTION

This paper presents the results of an experimental investigation of seismometers developed by the Bureau of Mines; these are especially adapted to measure vibrations caused by blasting or similar disturbances. The work represents an effort to develop a simple, lightweight, portable instrument that will measure these vibrations accurately. Several types of instruments were studied, and the theory of the instruments and the actual records made by them were correlated.

Seismometers in general may be classified in three groups—those that measure displacement, velocity, and acceleration. Each of these in turn may register the record by a mechanical system, a magnetic system, or an electrical system. The electrical system may be separated into two classes—one method using change of contact resistance and the other change of electrical capacitance.

The mechanical method, with its combined optical magnification system, in general lacks flexibility; a disturbance at various points is difficult to record simultaneously on the same sheet. Such instruments are also difficult to manipulate in the field, especially where more than one component of the vibrations is to be measured simultaneously.

The magnetic recorder is much simpler, can be easily damped, if properly designed does not require amplifiers, and offers the advantage of simultaneous recording. The sensitivity or range of motion of this recorder can be changed easily. One inherent disadvantage in the magnetic instrument is that the record is proportional not to displacement but to velocity of motion. Integration of the velocity of motion with an integration constant is therefore required to measure the actual relative displacements from which the stresses may be calculated.

The electrical contact-resistance method is better adapted to conditions in which the forces of acceleration are high and the background noises or vibrations due to passage of the currents in the various contacts of the actuating portions of the circuit are negligible.

The electrical capacitance method may be divided into two distinct branches. The first operates upon the change of frequency in an electrically oscillating system due to degree of approach and separation of two condenser plates which form the capacitance in an oscillatory circuit. This variable frequency is applied to the straight portion of a resonance curve in a coupled circuit adjusted to the proper constants. The method requires an oscillator and amplifying circuit which are inherently difficult to calibrate and keep in adjustment. The second system records the change of phase in two coupled oscillators as described in Bureau of Mines Technical Paper 518. Both capacitance methods have a wide range of sensitivity and are especially well-adapted for making measurements of this kind.

Instruments that measure acceleration have the disadvantage of requiring a double integration, with two integration constants, to determine the displacement and therefore have not been used where displacement values are desired.

In any study of vibrations an instrument capable of recording the actual displacement-time curve is highly desirable. Such a curve shows magnitude of displacement as a direct ordinate rather than an integration of an area, as in a velocity-time curve. If velocity and acceleration are desired they may be ascertained from the displace-

ment-time curve by measuring the slope at any point or the rate at which the slope is changing. For complete analysis of the motion displacement curves along three perpendicular axes must be obtained simultaneously and for convenience should be recorded on the same paper or film, one above the other. Of course, the two horizontal-component recorders will differ somewhat in design from the vertical-component instrument. The purpose of this paper is to explain the design of certain horizontal and vertical recorders and to describe tests of the vertical recorder on a mechanical oscillator,³ the motion of which is accurately known, so as to show the reliability and accuracy of the instrument.

ACKNOWLEDGMENTS

The author wishes to acknowledge his indebtedness to F. W. Lee, senior physicist, of the Bureau of Mines, for his direction of the work and for his many helpful suggestions, and to R. H. Baker, J. A. Rowe, and H. A. Dent, of the Pittsburgh Experiment Station, Bureau of Mines, for their assistance in construction of the apparatus.

SPECIFICATIONS

A seismic recorder should have the following characteristics:

1. The curve obtained should have an amplitude proportional to the actual motion for the entire range of frequencies the instrument will be required to record.
2. Transients and sustained oscillations should be recorded accurately.
3. The recorded wave and the actual vibration should be in phase at all frequencies within the working range of the instrument.
4. The instrument should be sensitive enough to record the microseisms constantly occurring in the earth. Higher sensitivity is not only useless but even objectionable.
5. Sensitivity should be adjustable, so that vibrations of large amplitude can be measured when the instrument is operating close to violent disturbances.
6. The calibration constant should be obtainable by simple and direct measurement in the field after the sensitivity has been adjusted to the correct value for the desired measurements.
7. The record should be permanent, distinct, and quickly obtained.
8. The apparatus must be portable.

All these requirements are met by the instruments developed in this investigation.

DESCRIPTION OF APPARATUS

The oscillograph used to record the vibrations was provided with 12 elements.⁴ Three components can be recorded at four points simultaneously on the same recording strip. The photographic attachment can handle sensitized paper or film up to 6 inches in width and about 100 feet in length. As recording is continuous while the motor is running short or long records can be made at will. Several gear ratios mounted on the motor control the speed. A battery-driven tuning fork of 100 vibrations per second operates an oscillating mirror so as to flash fine crosslines the entire width of the paper. An accurate time scale is thus provided, and at the same time the phase relations of the various curves on the paper⁵ can be readily compared. The apparatus is equipped with a cutting wheel so that the exposed portion of the film may be removed and developed whenever desirable.

³ Lee, F. W., and Irland, G. A., Construction of Master Mechanical Oscillator for Testing Seismic Recorders and Other Allied Apparatus: Tech. Paper 518, Bureau of Mines, 1932, 17 pp.

⁴ Built especially for the Bureau of Mines by O. S. Peters, Washington, D.C.

⁵ This attachment was added after most of the curves shown in this paper were made. The timing wave in these oscillograph curves was made by a vacuum-tube timing circuit.

A numbering stencil permits the operator to print identification numbers of 3 digits by setting 3 dials and opening a special shutter for several seconds. Figure 1 shows timing lines and identification number. A centimeter scale is mounted along the ground glass on which the light spots are focused, thus enabling the operator to adjust the zero position and sensitivity to suit the recording strip. The power for the light and motor may be supplied by a 6- or 12-volt storage battery. The oscillograph is mounted on top of a motor-driven developing tank, so that the record can be obtained in the field without a dark room.

THEORY OF OPERATION

In both horizontal and vertical seismometers a circular plate is supported by a very rigid arm attached to the frame by a flexible hinge. Two other condenser plates are mounted on the frame, with screw adjustments for varying the air gap on each side of the middle or

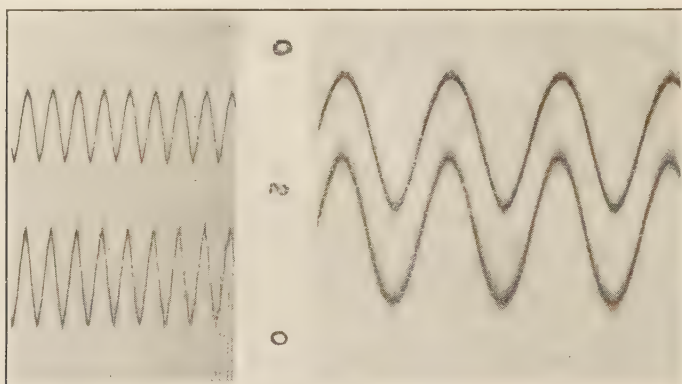


FIGURE 1.—Oscillograph record with timing lines and identification number.

hinged plate. In the horizontal instrument the axis of the hinge is slightly inclined from the vertical. The inclination of the axis thus imparts to the arm a slight restoring force due to gravity and causes it to oscillate with a very long period like a vertical pendulum of great length.

A vertical pendulum oscillates with a period given by the equation $T=2\pi\sqrt{\frac{L}{g}}$, where T is the period in seconds for one complete oscillation, L is the distance in centimeters from the point of support to the center of oscillation of the pendulum, and $g=980$ cm per second per second, the acceleration of gravity.

In this inclined pendulum the force that replaces g in the equation for the vertical pendulum is the component of gravity which acts in line with the pendulum, or $g \sin \alpha$ (fig. 2). Hence, the period is

$$T=2\pi\sqrt{\frac{L}{g \sin \alpha}}, \text{ or the natural frequency of oscillation, } \nu = \frac{1}{2\pi}\sqrt{\frac{g \sin \alpha}{L}}.$$

If $\alpha=\frac{\pi}{2}$, $\sin \alpha=1$ and the pendulum is a vertical one. If $\alpha=0$, $\sin \alpha=0$ and there will be no restoring force. If α is negative, the pendulum swings away from the central position and is unstable.

This theory is based on the assumption that the hinge does not exert any restoring force. For small values of α a very slight force exerted by the hinge may cause a large error in the period of oscillation computed from this formula. The restoring force of the hinge or suspension may be used instead of the component of gravity, in conjunction with it, or in opposition to it.⁶ In the horizontal seismometer described in this paper the hinges are very flexible, and a component of gravity from a small inclination (α) provides the chief restoring force. In vertical-component seismometers the necessity of supporting the

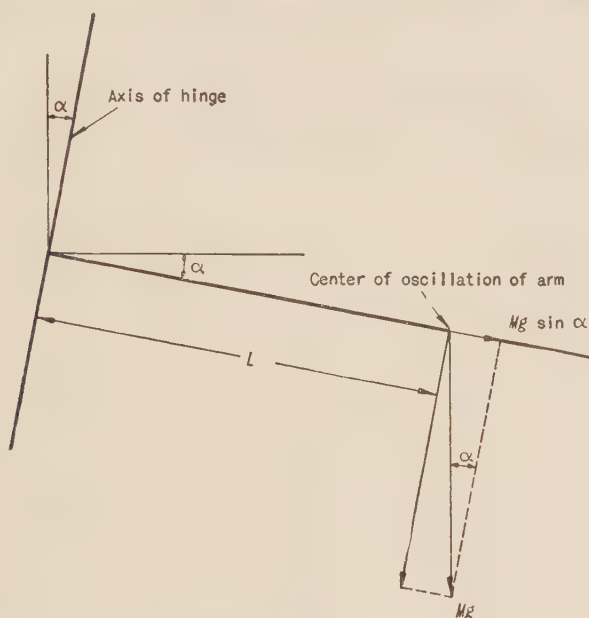


FIGURE 2.—Diagram of inclined horizontal pendulum.

mass of the hinged arm against the force of gravity entails additional complication. This phase of the subject is discussed later.

In the method of recording that depends upon the change of capacitance between plates the charge on the plates exerts a slight attractive force. In the neutral position this force is divided equally between the two and produces no unbalancing action, but when the middle plate moves closer to one of the outer plates the attraction toward that plate increases and toward the other decreases. Unless the restoring force is sufficient to overcome this force the plates will swing until they come in contact and will then hold together until the circuit providing the charge is opened. If the plates are far apart the force exerted by the charge is weak and the restoring force need not be strong, but if they are set close together for high sensitivity a greater restoring force is necessary. If they are set for the highest sensitivity required the natural frequency of the arm may still be kept well below the lowest frequency the instrument is required to measure.

⁶ Anderson, J. A., and Wood, H. O., Description and Theory of the Torsion Seismometer: Bull. Seismol. Soc. America, vol. 15, March 1925, p. 9.

As shown in the mathematical analysis given in the appendix, critical damping is necessary for satisfactory results. The instrument may be effectively damped by attaching a copper vane to the end of

the hinged arm so that it will swing through the field of a permanent magnet. This method is the same as that used in many electrical meters. The damping may be adjusted by slight changes in the air gap, made by placing shims between the ends of the magnet and the pole pieces, or part

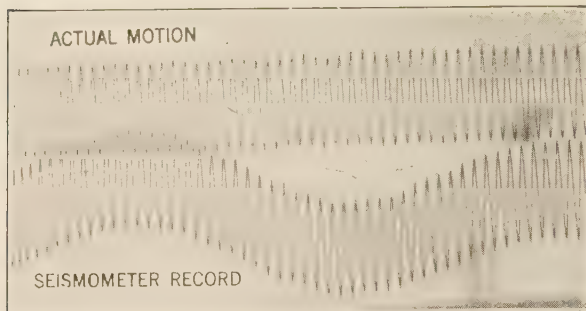


FIGURE 3.—Oscillograph record of underdamped seismometer. Applied frequency of vibration, 25 cycles per second; natural frequency of instrument, 0.7 cycle per second.

of the magnetic flux may be bypassed by an adjustable magnetic shunt. If the instrument is underdamped the axis of the curve will often swing at the natural frequency of the arm, as shown in figure 3.

When the seismometer vibrates the center of oscillation moves slightly. The appendix gives the development of the equation of this movement for sinusoidal vibration suddenly applied to a critically damped seismometer. This equation (16) is the fundamental basis of design used for constructing the successful instruments described in this paper. It is used to determine from the design constants of the instrument how much the steady mass moves under

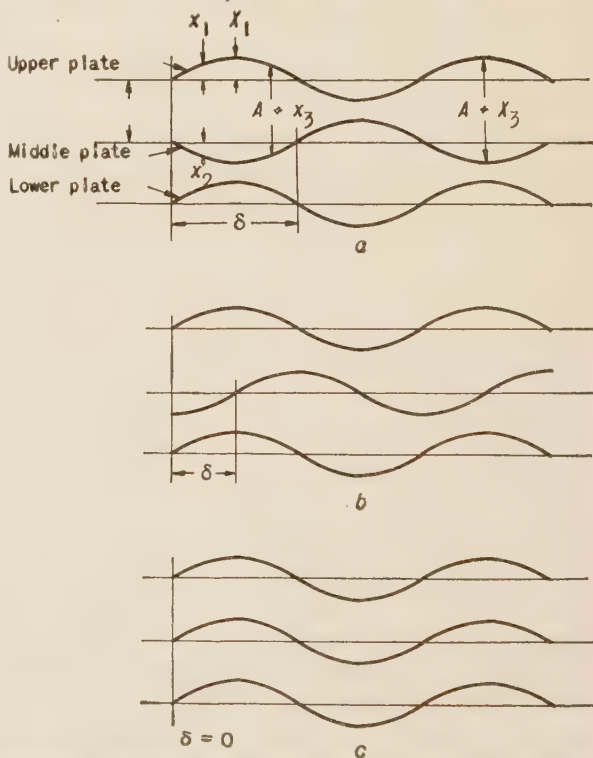


FIGURE 4.—Diagram showing relative motion of middle and outer plates for different phase angles.

different conditions of operation and to show how certain changes in design affect the accuracy of the instrument. It also may be used to show that a seismometer can be designed in which the center of

oscillation moves a negligible amount over a considerable range of frequency.

ERROR DUE TO MOVEMENT OF MIDDLE PLATE

To record accurately the displacement of the seismometer the middle plate must remain fixed in space; if it vibrates appreciably an error in the record will result. Not only must the magnitude and phase of the displacement of the center of oscillation as given by equation (16) be studied, but also the effect of this movement on the recorded curve must be determined. Figure 4 shows the motion of the 3 condenser plates for 3 values of phase angle δ . It also shows plainly that in *a* the

amplitude of the record is increased by the motion of the middle plate; in *b* the amplitude is not greatly affected, but the phase is shifted; and in *c* the record will be a straight line even though the instrument has considerable vibration. If the amplitude of the middle plate had been less than that of the outer plates, as in *c*, the curve would have been in phase with the displacement of the instrument but greatly reduced in amplitude. If the vibration of the middle plate had been in phase with the displacement of the instrument, as in *c*, but of greater amplitude the record would be opposite in phase to the actual vibration of the instrument.

This effect is expressed mathematically in the appendix, equations (26) and (27). With these the error in magnitude and phase for any frequency can be computed from the design constants of the seismometer. Values were measured for a sample instrument and errors com-

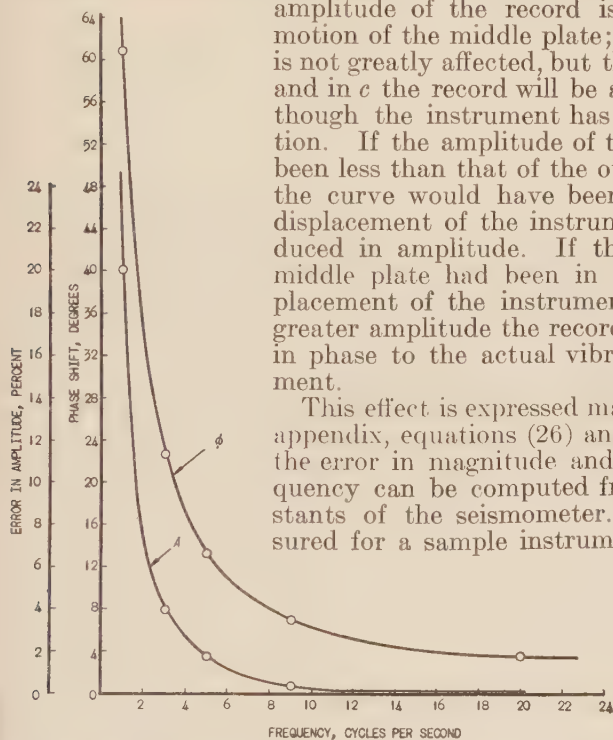


FIGURE 5.—Error in amplitude and phase plotted against frequency: *A*, Amplitude; ϕ , phase.

puted. (See table 1, appendix, and fig. 5.) The curves of figure 5 show that an instrument of this design would have an error in magnitude of less than 6 percent for frequencies above 2 cycles per second and an error of less than 35° in phase angle above this frequency. If the lower limit of frequency is too high for satisfactory operation it may be reduced by decreasing the restoring force. In the horizontal seismometer the restoring force can easily be decreased by adjusting the leveling screws so as to reduce the angle of inclination α . Other means would be necessary in the vertical instrument. If it is found that the restoring force is as low as can be allowed more radical changes in design are necessary to increase L or M or both. These computations and curves are given merely to illustrate the method used in checking the performance of a proposed design. After construction the seismometer should be tested on the mechanical oscillator at various frequencies to make sure that errors are not produced by unforeseen causes.

REVERSAL OF PHASE

If the natural frequency of the hinged arm is so high that it comes within the required range of the instrument the errors become very great, and the error in phase, angle ϕ , may approach 180° . (See table 2, appendix, and fig. 24.) The computations and curves show that the phase angle ϕ between the actual motion and the recorded curve passes through 90° at about 2.8 cycles per second and below that frequency approaches 180° . The amplitude below this frequency becomes negligible, and the instrument fails to record because the middle plates move virtually the same amount as the frame and in phase with it.

The theoretical computations and curves apply to a critically damped seismometer. If an instrument is underdamped the middle plate in passing through this critical frequency, and for some cycles above and below it, may vibrate several times as much as the frame. If it vibrates more than the frame at a frequency below the critical

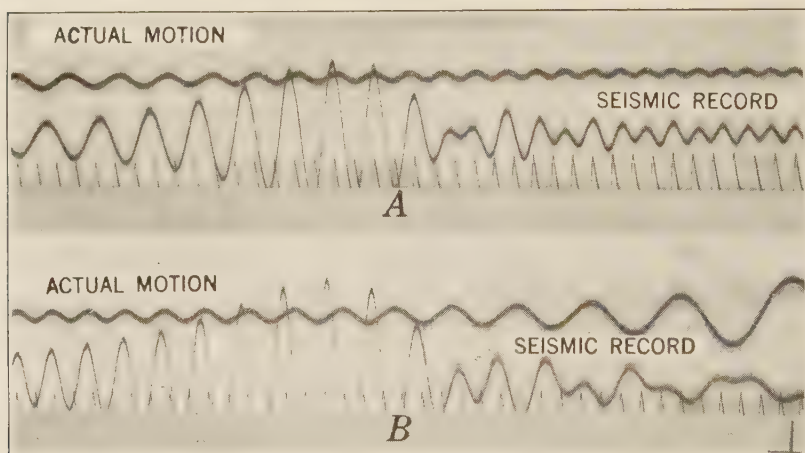


FIGURE 6.—Oscillograph curves showing phase-reversal effect below critical frequency: *A*, Frequency increasing; *B*, frequency decreasing.

value the recorded curve will appear virtually 180° out of phase with the curve of motion. This effect is shown in the oscillograph records of figure 6. In *A* the frequency of the master oscillator was raised gradually, and in *B* it was lowered. The upper curve shows the actual motion as obtained from the condenser plates mounted on the moving table and stationary frame. The lower curve is the record obtained from the seismometer placed on the table. A timing wave of 12 cycles per second is shown at the bottom of the record. The driving force of the table was kept constant, and the amplitude of the table motion decreased or increased as the frequency was raised or lowered owing to the mechanical impedance of the table increasing at higher frequencies. The record shows plainly that the phase is reversed below the critical frequency, passes through 90° at that value, and comes asymptotically into phase at frequencies above the critical value. The large amplitude of the curve at the critical frequency is caused by the maximum amplitude of the motion of the

middle plate, which is out of phase with the outer plates. The irregular waves occurring just after the critical value in both records are caused by a transient effect. The middle plate continues to swing at its own natural frequency, while the applied frequency is changing until the energy it accumulates is dissipated by the damping. The result is an irregular wave produced by interference between two slightly different frequencies.

FLEXIBILITY OF ARM

The theory of the seismometer has been developed on the assumption that the arm is perfectly rigid and indicates that the error becomes negligible as the frequency becomes high. If the arm carrying the steady mass is not stiff enough its natural frequency when vibrating as a reed may come within the desired useful range of the instrument. In this event the middle plate vibrates because of the bending of the arm, and the instrument acts as it does at low frequency when the natural frequency of the arm is approached. Figure 7 shows an oscillograph record of an instrument with a flexible arm, which

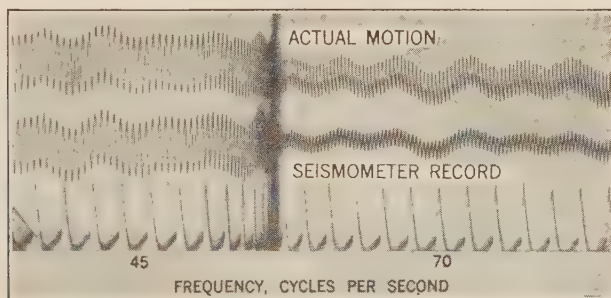


FIGURE 7.—Oscillograph curves showing error caused by bending of arm, compensated-spring-type vertical recorder. Error at 70 cycles caused by flexibility of arm carrying middle plate.

operated satisfactorily at 45 cycles and below but showed violent vibrations of the middle plate at 70 cycles per second. The table curve shows some distortion due to reflection of vibration of the steady mass back to the table, giving a "beat effect" like that observed in closely coupled electrical circuits.⁷ This difficulty can be overcome by making the arm rigid enough to withstand the greatest bending forces that can occur within the frequency range of the instrument. A thick aluminum arm with side braces eliminates this trouble without excessive weight.

LOCATION OF CENTER OF OSCILLATION

In computing the error caused by vibration of the center of oscillation of the arm the middle plate was assumed to remain parallel to the outer plates and the axis through the center of oscillation to coincide with the center line of the circular condenser plate on the hinged arm. As neither of these assumptions is necessarily correct the error produced must be determined when these assumptions are not in agreement with the facts.

⁷ Morecroft, J. H., *Principles of Radio Communication*: John Wiley & Sons, Inc., 2d ed., 1927, chap. 3, p. 279.

Obviously, if the center of oscillation remains at rest and coincides with the center of the circular plate this plate must rotate about its center line as the frame is raised and lowered by the motion of its base. Rotation of the middle plate about its center line will cause a net increase of capacitance between it and either outer plate, as the increase in the half of the plate that approaches is greater than the decrease in the half that recedes. However, if the axis of the middle plate is halfway between the two outer plates—it should be so adjusted—the increase on both sides will be equal. As these two condensers are connected in a push-pull circuit the equal increases in the two values of capacitance neutralize each other, and no change occurs in the oscillograph record due to rotation alone. If the middle plate is not exactly halfway between the two outer plates an error will result, but as the rotation is always very small and the middle plate is always nearly equidistant from the outer plates this error will be negligible.

If the center line of the condenser plate does not coincide with the center of oscillation of the arm the middle plate will rotate about an axis other than its center line while the instrument is vibrating, even though the center of oscillation should remain fixed in space. If the center of oscillation is farther out than the center line of the plate the sensitivity of the instrument is reduced, and if its radius is shorter than the center of the plate the sensitivity is increased. Rotation about an axis other than the center line may be separated into two component motions which are equivalent to this motion: (1) Rotation about the center line and (2) parallel translation of the plate. As was just shown, rotation about the center line has little or no effect on the record obtained from the instrument. Parallel translation of the middle plate either increases or decreases the record, depending upon whether the motion is in phase or opposite in phase relative to the motion of the frame. Whether this change in sensitivity is an increase or decrease depends upon the relative distances from the axis of the hinge to the center line of the plate and to the center of oscillation. The ratio of these two distances is fixed and can be taken care of in the calibration constant. Motion of the center of oscillation is then the only cause of error, and this error will agree with the value found for it by theoretical computations.

As stated, the sensitivity can be increased by designing the instrument so that the center line of the condenser plate is farther from the axis of the hinge than is the center of oscillation. This method of obtaining high sensitivity is not desirable, as it gives L a relatively small value. The value of L should be kept large so that the natural frequency of the arm will be low. (See formula for natural frequency of arm.) Sensitivity can generally be more effectively increased by designing the instrument with larger condenser plates rather than shorter radius to the center of oscillation.

DESCRIPTION OF HORIZONTAL-COMPONENT SEISMOMETER

Figure 8 shows the horizontal-component seismometer developed during this investigation. Hinged arm a is made of aluminum three-sixteenths inch thick, cut out where possible to reduce weight, and stiffened by side braces of the same material one-half inch wide. The hinge is mounted on bakelite support b , which insulates it from the frame. Hinge c consists of 4 crossed strips of thin bronze, 2 at

the top and 2 at the bottom. The weight of the arm holds the strips taut. The necessary restoring force is provided by raising the hinged end slightly higher than the other end by means of leveling screws *d*. The restoring force can thus be adjusted by changing the inclination. Outer condenser plates *e* are made of brass and are mounted on the frame by bakelite brackets *f*. They are carried on fine screws with graduated knobs so that the air gap may be adjusted to different widths for varying sensitivity. Each graduation indicates a movement of the plate of one-thousandth of an inch. Electrical connections to the three condenser plates are made at binding posts *g*. Contact with the middle plate is made by the bronze hinge strips and to the outer plates by wipers *h* against the graduated knobs. Copper plate *i* on the end of the hinged arm passes between pole pieces *j* of magnets *k* to provide damping. The instrument is calibrated by turning down graduated screw *l* until it touches the copper damping vane, then releasing wire spring *m*, which holds the arm

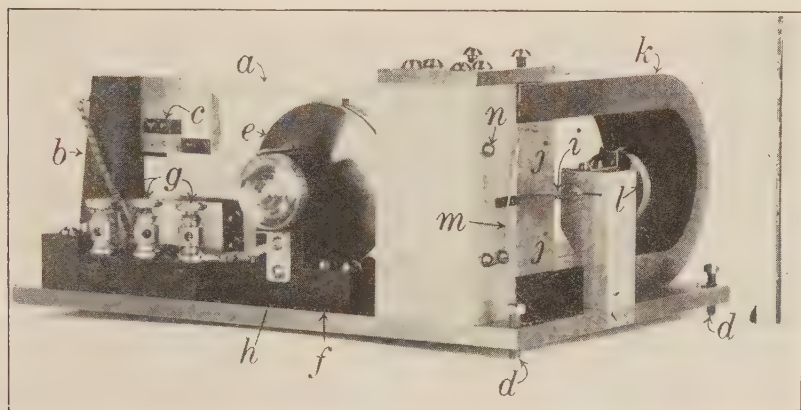


FIGURE 8.—Horizontal-component seismometer.

firmly against the calibrating screw. The arm can then be moved any known amount and the corresponding movement of the light spot on the oscillograph noted. After calibration spring *m* is secured behind catch *n*, which holds it away from the arm, and the screw is backed away enough to avoid interference with the movement of the arm while recording.

THEORY OF VERTICAL-COMPONENT SEISMOMETER

In the horizontal-component seismometer the force of gravity is used as a restoring force only; it exerts the same force for a given displacement on either side of the neutral position. In the vertical-component seismometer, on the other hand, gravity tends to pull the hinged arm away from the required neutral position, so that some upward force must be provided to support the weight. If this upward force is supplied by a helical spring pulling at right angles to the arm to which it is attached the arm tends to oscillate about a neutral position, where the pull of the spring and the attraction of gravity are exactly equal. When the arm is supported thus the spring also tends to vibrate transversely at certain critical frequencies. (See appendix, p. 41.) The natural frequency of the spring for

transverse vibrations is always high compared to the desired natural frequency of the arm. As transverse vibrations of the spring cause vibration of the arm it supports the instrument must be so designed that there will be very little spring vibration at all frequencies within the useful range of the seismometer. Only a spring having a very high fundamental natural frequency will serve this purpose. Transverse vibrations occur not only when the spring is subjected to frequencies near its natural frequency but also for fractions of its natural frequency, given by the harmonic sequence $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. (This effect can be seen plainly in figs. 9 and 11.)

The design problem is to make the natural frequency of the arm well below the frequency range of the instrument and to make the natural frequency of the spring for transverse vibrations far above the upper limit of frequency range of the instrument. Unfortunately, changes in spring design which tend to lower the natural frequency of the arm also lower the natural frequency for transverse vibrations. Increasing the mass of the arm or shortening the radius arm to the point where the spring is attached, although tending to lower the natural frequency of the arm, necessitates the use of a stronger spring and so lowers its natural frequency for transverse vibrations. Table 3, appendix, shows that this method of support gives a very short range of satisfactory operation, regardless of the selection of design constants.

This difficulty can be overcome by using a long spring to obtain a low natural frequency for the arm and mounting damping springs or snubbers along the main spring to reduce the transverse vibrations. Such an instrument would be satisfactory at low frequencies, but at even moderately high frequencies would be difficult to damp enough to give satisfactory operation.

FLYWHEEL-TYPE VERTICAL-COMPONENT SEISMOMETER

Adding mass which must be supported by the spring necessitates use of a stronger spring and so counteracts the gain (see table 3). However, if mass is added to the arm so that it does not add to the pull of the spring the frequency of the arm can be lowered without affecting the natural frequency of the spring for transverse vibrations. To obtain this result half the weight can be added on each side of the point of attachment to the frame. A simple way to apply mass is in the form of a flywheel with its center at the axis of the hinge. As the weight of the flywheel is balanced it does not require any lifting force, and the spring will be required to support only the arm. With this type of construction a light spring—one having a small constant, k —can be used. The combination of large mass and a light spring gives a low value for the natural frequency of oscillation of the arm, and at the same time n_s , the transverse vibration frequency of the spring, will be high. This construction, however, has three disadvantages: (1) The flywheel increases the size and weight of the instrument; (2) the large mass of the flywheel makes critical damping of the arm much more difficult than for a lighter arm; and (3) the mass on the opposite side of the hinge has an inertia effect opposing that of the arm and tending to oppose rotational acceleration of the arm. This tendency increases with higher frequencies, thus reducing the sensitivity and making it impossible for one calibration to hold for all frequencies within the range of the instrument.

In figure 9 the decrease in sensitivity with increase of frequency is quite evident.

Even when the arm is loaded and the frequency of the spring raised to a high value the spring still tends to vibrate at harmonic fractions

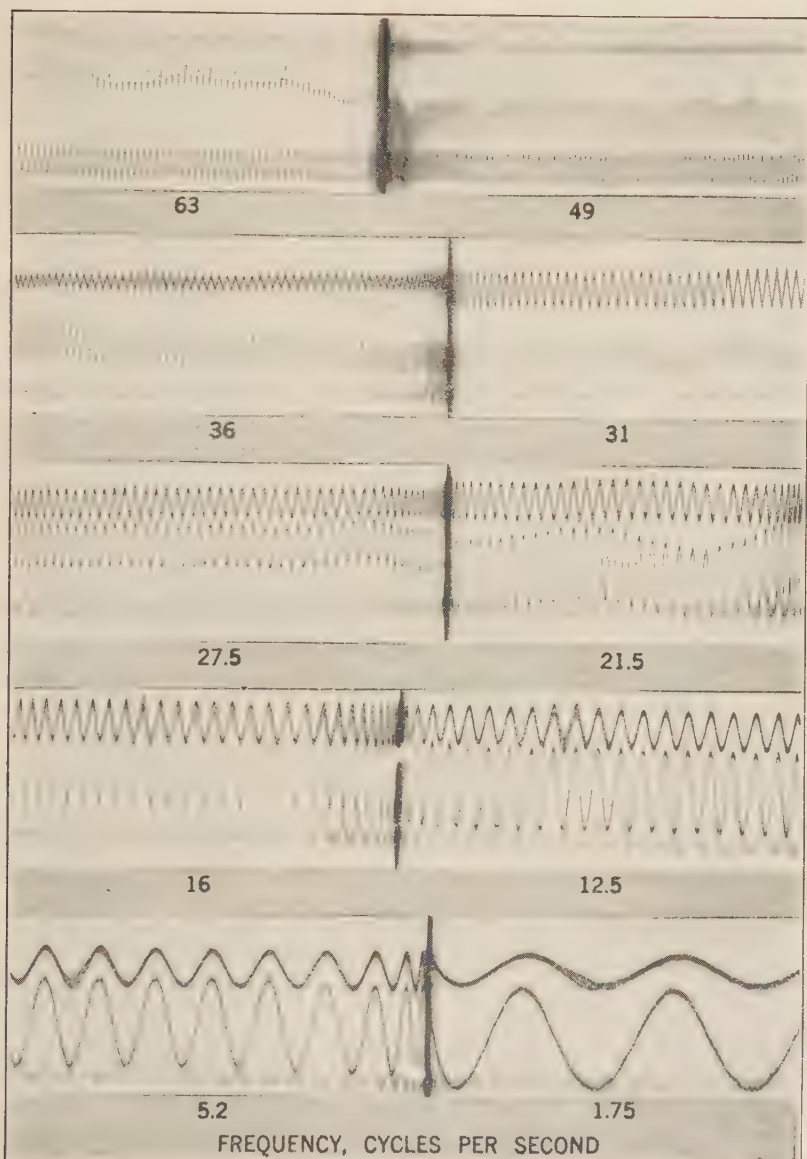


FIGURE 9.—Oscillograph records from flywheel-type vertical-component seismometer, showing effect of spring vibrations at critical frequencies.

of its natural frequency. In figure 9 harmonics due to spring vibration are very pronounced at 63 and 31 cycles per second, indicating a spring frequency of some multiple of 63 cycles. The slight ripple in the curve at low frequencies appears to have a frequency of about

63 cycles per second. Measurements were made of this spring, and its natural frequency for transverse vibrations was computed at 52 cycles per second. End effects due to the method of attaching the spring to the instrument were probably responsible for the failure to check more closely the value observed in the oscillograph record.

The flywheel-type seismometer used in making the records (fig. 9) is shown in figure 10. Arm *a* is attached to the frame by hinges *b*, which consist of crossed bronze strips. Heavy brass flywheel *c* is used to load the arm. Magnet *d* provides magnetic damping, similar to that used in the Wenner instrument; pole pieces *e* are designed to produce a radial field in which moves a cylindrical coil attached to the arm. Helical spring *f*, which supports the weight of the arm, can be adjusted by means of screw *g*. Outer plates *h* have a screw adjustment to vary the sensitivity. Micrometer screw *i* is used for cali-

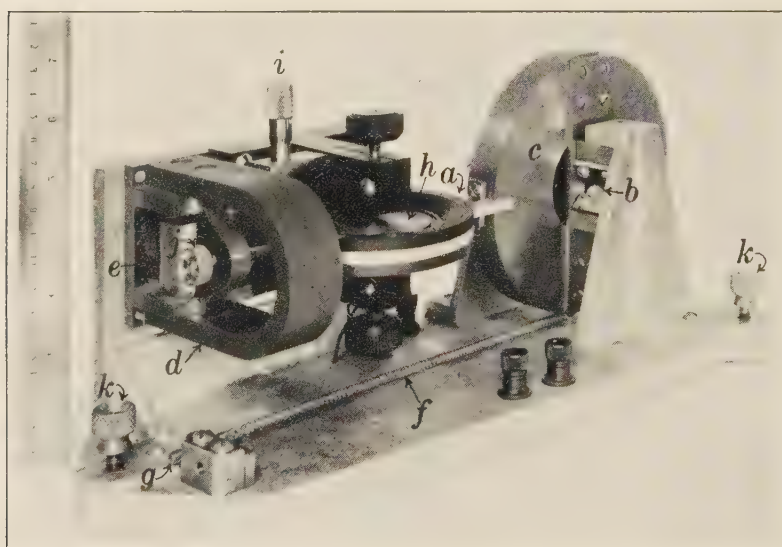


FIGURE 10.—Flywheel-type vertical-component seismometer.

bration. Screw *j* is used to clamp the arm for carrying. The frame is an aluminum casting and is leveled by screws *k*.

SOLENOID-TYPE VERTICAL-COMPONENT SEISMOMETER

To obtain a vertical-component seismometer free from the difficulties just considered, the arm must be supported in some other manner. The ideal method is a system in which the constant upward pull just equals the downward pull of gravity. An open-core solenoid has a constant pull when its plunger is inserted more than 40 percent and less than 80 percent of its length. This method of support was tried, using a weak spring for the restoring force. When the instrument was properly adjusted the performance was good, but this method was abandoned because of the difficulty of maintaining correct adjustment. The value of current necessary to provide the correct pull was so critical that slight changes of temperature in the coil or voltage of the battery caused excessive shifting of the zero position. Moreover, a storage battery and control apparatus must be provided.

SPRING COMPENSATION

Another method of obtaining a virtually constant upward pull is to use a spring, which will provide the necessary force, and some means of compensating for change in spring pull due to change in length by some other force, which will aid the spring and will change an equal amount in the opposite direction. A permanent magnet having specially shaped pole pieces and armature might be used to aid the spring so that the magnetic pull would increase at the same rate that the spring pull decreases. This method was not attempted although some experiments were made which indicated that a magnetic device with the proper characteristics could be constructed.

A counterweight mounted on an arm slightly inclined to the vertical can also be used to compensate for change in force of the spring. (See fig. 11.) Here the compensation is due to the change in effective radius of the moment of the counterweight as the arm rotates through a small angle. By proper design this method can be made

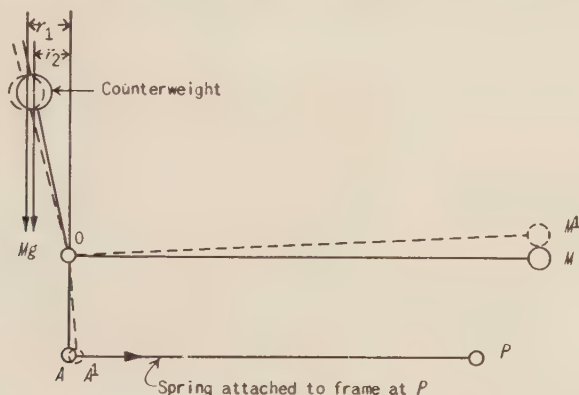


FIGURE 11.—Counterweight method of compensating for change in force exerted by spring.

to give exact compensation. It can be used, however, only when no components of vibration exist in the horizontal plane and perpendicular to the axis of the hinge. For such vibrations the counterweight acts as a steady mass and causes the instrument to record a combination vertical and horizontal vibration.

Figure 12 shows oscillograph records taken on the mechanical oscilator with an instrument of this type. Here the vibrations were restricted to the vertical direction, and no error was produced by the counterweight. The effect of spring vibration is quite evident at certain frequencies, and wavering of the axis due to underdamping is apparent.

Figure 13 shows almost perfect agreement between the actual motion as recorded by the table condenser plates and the curve of the recorder. The upper curve shows the actual motion and the lower curve the record of the instrument. These curves were taken by the same instrument that was used for figure 12 except that rubber bands were applied as snubbers to damp the spring vibrations and critical damping was obtained by increasing the field.

GEOMETRICAL COMPENSATION

By applying the spring in various ways, one of which is shown in figure 25, the spring does not act directly to lift the arm but rather to produce a moment equal to that of the arm due to gravity. For

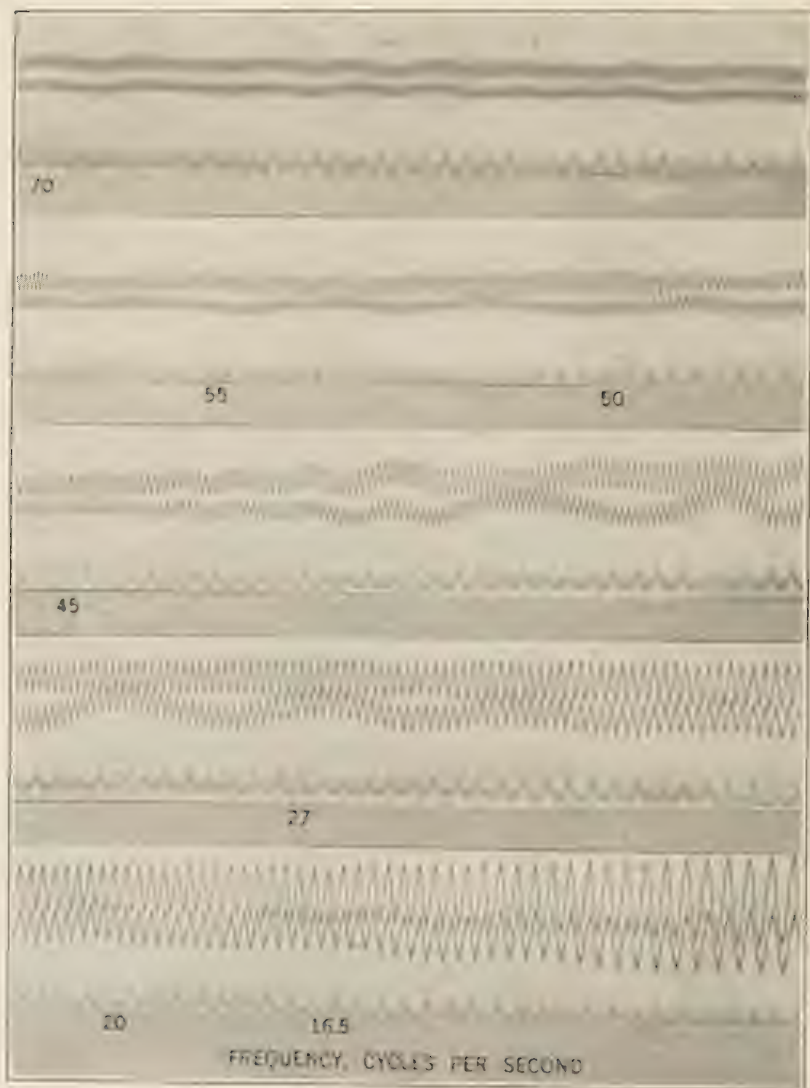


FIGURE 12.—Oscillograph records from compensated-spring-type vertical-component seismometer with long spring, showing effect of spring vibration at various frequencies. Upper curve shows motion, lower curve shows seismometer record.

small movements of the arm the change in effective radius of its mass is negligible, and the moment due to gravity may be considered constant. The change in pull of the spring due to change in its length is accompanied by a change in its effective radius, so that the product of the two—the moment produced by the spring—remains nearly

constant.⁸ As shown in the appendix (p. 44), the dimensions can be adjusted so that the compensation is exact for small angles of rotation. When the spring is overcompensated the arm is unstable and swings away from the neutral position in either direction it may be started. When the compensation is exact the arm will remain in any position near the neutral position in which it may be placed. When the spring is slightly undercompensated it has a slight restoring force, and the arm tends to oscillate at low frequency. As may readily be seen, the behavior of this instrument is just the same as that of the

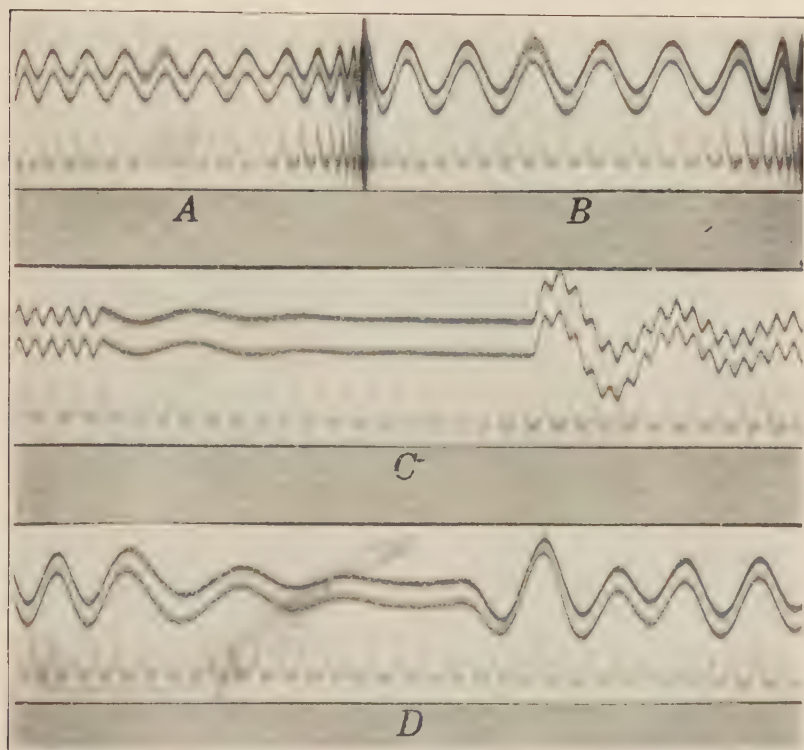


FIGURE 13.—Oscillograph records of transients, vertical-component scismometer with long spring compensated by counterweight. Arm damped by electromagnet; spring damped by snubbers. Upper curve shows actual motion; lower curve shows scismometer record. *A*, 7 cycles per second; amplitude of table, 9.65 microns (0.00038 inch). *B*, 4 cycles per second; amplitude of table, 19.05 microns (0.00075 inch). *C*, Transient, opening and closing a-c. circuit, 18 cycles per second. *D*, Transient, opening and closing a-c. circuit, 4 cycles per second.

horizontal pendulum when the angle of inclination is negative, zero, or slightly positive.

By making the spring very short the natural frequency of the spring for transverse vibrations may be made high enough to avoid trouble. The value for spring constant k will therefore be large, but as the equation for the natural frequency of the arm no longer applies this causes no trouble. The natural frequency of the arm now depends only on the degree of compensation and may be made as low as desired, regardless of the value of k or M . Design computations for an instrument of this type are given in the appendix (p. 45).

⁸ Hort, W., *Technische Schwingungslehre*: Julius Springer, Berlin, 1922, p. 81.

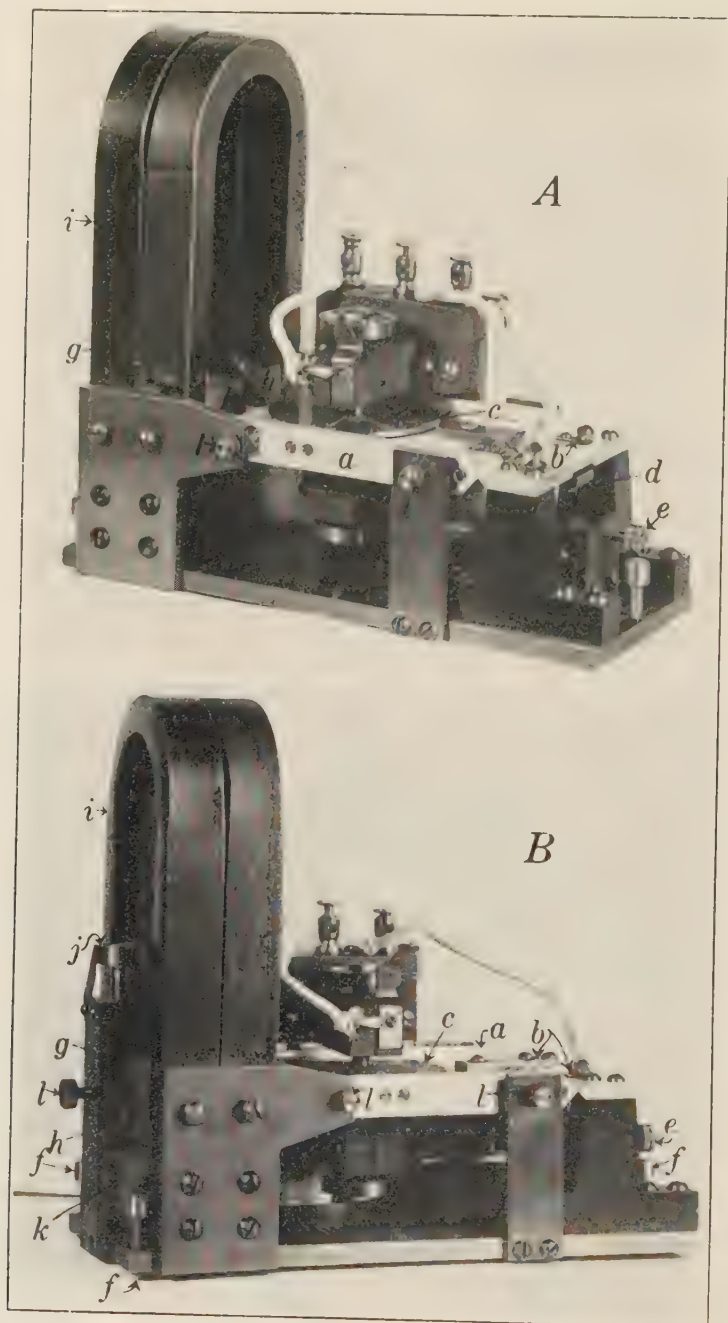


FIGURE 14.—Geometrically compensated type showing: A, Adjusting screw; B, calibrating device.

DESCRIPTION OF COMPENSATED-SPRING-TYPE SEISMOMETER

Figure 14, *A* and *B*, shows two views of a compensated-spring-type vertical-component seismometer. A rigid aluminum arm, *a*, is attached to the frame by four bronze strips, *b*. Outer condenser plates *c* are adjusted as in the horizontal instrument. The weight of the hinged arm is supported by small helical spring *d*, the tension of which is adjusted by screw *e* so that the arm swings freely midway between the outer plates. Spring compensation can be finally adjusted when the instrument is assembled by cutting off small amounts from the end of the spring and trying the spring in place until there is just enough restoring force. A few days are required to season the spring before it will hold its adjustment. It can then be kept in adjustment with little attention. The base plate is leveled by means of screws *f*. Copper damping vane *g* passes between the two pole pieces *h* of magnets *i*. Screw *j* and wire spring *k* are used for calibrating, as on the horizontal instrument. Screws *l* are used for clamping the arm when the instrument is being carried about.

METHOD OF CALIBRATION

One great advantage of the condenser-type recording instrument is the ease with which the apparatus may be calibrated in the field. After the outer plates have been adjusted to the spacing desired for the amplitude of vibration to be measured the instrument can be calibrated for that spacing by means of a micrometer calibrating screw mounted on the frame. The middle plate, swinging freely, is adjusted about at the midposition between the two outer plates, and the oscillator trimming condensers are adjusted to give about 3 milliamperes through the oscillograph, as indicated on the milliammeter of the oscillator set. The position of the light spot on the ground glass of the oscillograph is then adjusted to the proper zero position for the curve to be recorded, using the galvanometer adjustment of the oscillograph. Its position is then read on the scale mounted along the ground glass. Next the calibrating screw is turned until it touches the arm, and the wire spring that presses the arm against the calibrating screw is then released. When the arm is thus clamped the screw is carefully adjusted so that the light spot of the oscillograph takes the same position it had when the arm was swinging freely. The calibrating screw may then be turned as many thousandths of an inch each way as desired and the corresponding movement of the light spot read on the oscillograph scale. The reading of the micrometer calibrating screw must be corrected by multiplying it by the ratio $\frac{\text{radius to the center of oscillation}}{\text{radius to the calibrating screw}}$. This ratio must be known for the instrument. The distance the light spot shifts when the center of oscillation of the arm moves one-thousandth inch may be taken as the calibration constant of the instrument adjusted as it was when the calibration was made. If the calibration is desired in microns the correction ratio may be multiplied by a conversion factor, and the new ratio used. Any point on the recorded wave may be measured from the zero position, and its distance divided by the calibration constant will give the displacement at the instant considered. After the calibration has been completed the pressure spring is pushed back from the arm and secured, the microm-

eter screw is backed away from the arm, and the instrument is ready for use.

The correction factor may be found in two ways. One method is to find the radius to the center of oscillation experimentally, measure the radius to the calibrating screw, and compute the ratio directly. The radius to the center of oscillation may be found before the instrument is completely assembled by clamping the hinge securely and allowing the arm to hang freely as a vertical pendulum. The arm is then set to swinging and its frequency carefully timed. From this

frequency and the formula $\nu = \frac{1}{2\pi} \sqrt{\frac{g}{L}}$ the value of L may be found.

This value is the radius to the center of oscillation of the arm.

The other method of finding the correction ratio is by means of the mechanical oscillator. The instrument is calibrated in the usual manner, assuming a correction ratio of unity. It is then placed on the mechanical oscillator, which has previously been calibrated against the optical measuring system or the table condenser plates. Vibrations of various amplitudes and frequencies are then applied to the instrument. The ratio of the amplitude recorded by the instrument to the true amplitude as recorded by the condenser plates of the mechanical oscillator will be the correction factor of the instrument. This should be constant for all tests, and an average of several tests should be taken. The vertical-component seismometer shown in figure 14 was tested on the mechanical oscillator with the following results:

Frequency, cycles per second	Amplitude from table condenser plates, inch	Amplitude from seis- mometer, inch	Correction- factor ratio
15	0.000313	0.000581	0.539
10	.000502	.000957	.524
			1.532

¹ Mean.

A rough check on the correction factor for this same instrument computed from experimental determination of the distance from the axis of the hinge to the center of oscillation was:

$$\frac{\text{Distance to center of oscillation}}{\text{Distance to center of calibrating screw}} = \frac{8.63}{14.78} = 0.583.$$

The difference between these two results is less than 10 percent.

TESTS OF VERTICAL-COMPONENT SEISMOMETER ON MECHANICAL OSCILLATOR

The vertical-component seismometer shown in figure 14 was tested on the mechanical oscillator at frequencies of 1.8 to 70 cycles per second. Oscillograph curves of various frequencies are shown in figure 15; the lower curve shows the instrument record and the upper curve the record made by the condenser plates attached to the table. The distortion occurring at 1.8 cycles is caused by movement of the center of oscillation as the frequency approached the natural frequency of the arm, about 0.7 cycle. This swinging of the arm

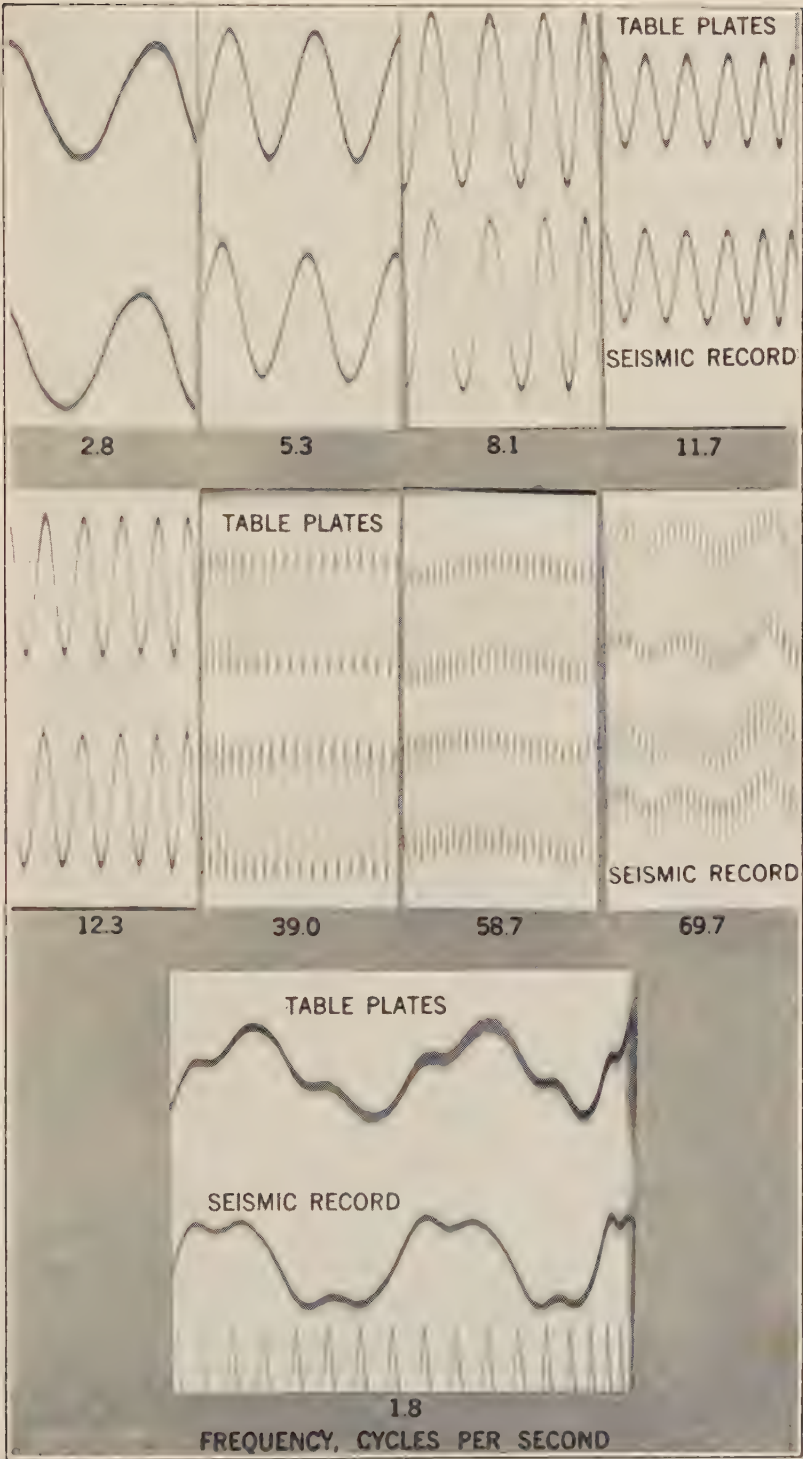


FIGURE 15.—Steady-state vertical vibrations of various frequencies.

caused variation of the effective mass of the table, and the distortion was reflected to the table motion. Examination of the wave shapes at this frequency shows that the third harmonic waves of the two curves are in phase, while the fundamental waves are about 60° out of phase. This difference may be explained by the fact that the instrument has little phase error for vibrations of 5 cycles but considerable for waves of 1.8 cycles.

Figure 16 shows curves plotted from the last two columns of the table that follows.

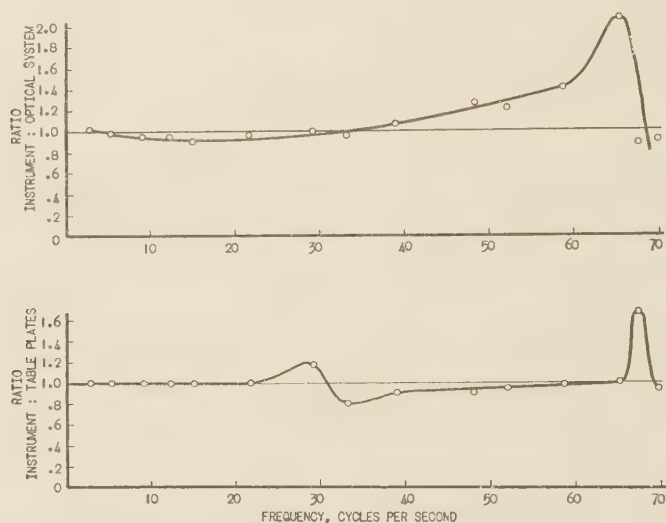


FIGURE 16.—Curves showing accuracy plotted against frequency.

Frequency, cycles per second	Optical scale, cen- timeters	Table curve, inches	Instrument curve, inches	Ratio, column 4 $10 \times$ column 2	Ratio, column 5 av. of column 5	Ratio, column 4 column 3
1	2	3	4	5	6	7
2.8	5.5	0.86	0.86	1.56	1.11	1.0
5.3	7.0	.96	.96	1.37	.98	1.0
8.1	9.5	1.26	1.26	1.33	.95	1.0
12.3	7.5	1.00	1.00	1.33	.95	1.0
15.0	8.0	1.08	1.08	1.25	.89	1.0
21.7	7.0	.94	.94	1.34	.96	1.0
29.3	6.5	.86	.92	1.41	1.01	1.17
33.2	5.5	.92	.74	1.35	.96	.80
39.0	6.0	1.00	.90	1.50	1.07	.90
48.2	5.0	1.00	.88	1.76	1.26	.88
52.0	5.5	1.00	.94	1.71	1.22	.94
58.7	4.5	.90	.88	1.95	1.39	.98
65.3	5.0	1.42	1.44	2.90	2.07	1.01
67.5	6.0	.50	.74	1.23	.88	1.48
69.7	7.0	.94	.88	1.26	.90	.94

The error around 30 cycles was due to vibrations of the condenser plates on the calibrating table and not to the error in the seismometer record. The optical measurements probably were not very dependable at the higher frequencies. At some frequencies the width of the light beam was very difficult to determine because the beam jumped up and down at about the natural frequency of the table. (See oscillograph record for 69.7 cycles, fig. 15.)

Tests by Rockwell⁹ show that the frequency of earth vibrations caused by blasting is 10 to 15 cycles per second. As the seismometer just discussed has a high degree of accuracy for this range of frequency it should be well suited for investigation of disturbances of this nature.

EXPERIMENTAL TEST FOR ACCURACY DURING TRANSIENT CONDITIONS

A seismometer may respond accurately to sustained oscillations of various frequencies and of complex wave shape but during the first few cycles after the vibration begins may record inaccurately owing to transient effects in the instrument. As seismic disturbances consist of rapidly succeeding wave trains the ability to record accurately the beginning of each new disturbance, even though it may be superimposed upon other vibrations then active, is just as essential for a successful seismometer as ability to record accurately after transient effects in the instrument have subsided. Whether the instrument ceases to record as soon as the disturbance ends or whether, owing to storage of energy in the instrument, it continues to indicate vibrations after the earth has come to rest must be determined. Computation of these transient effects mathematically is extremely difficult if not impossible. (See appendix, equation (16), p. 33.) It is therefore important to determine experimentally on the master oscillator how faithfully the seismometer records the beginning of a suddenly applied oscillation or the ending of a disturbance.

To test the seismometer for transient effects the curves in figure 17 were taken. Figure 17, *A*, shows a steady vibration of 25 cycles per second. This was suddenly removed, and a 4-cycle damped wave of small amplitude was substituted. Just as this died out the 25-cycle vibration, superimposed on a 4-cycle damped wave, was applied. In figure 17, *B*, a steady vibration of 10 cycles was applied throughout the whole time, and at the middle of the record a 4-cycle damped wave was superimposed on the steady vibration. Figure 17, *C*, shows a 4-cycle damped vibration applied to the table while at rest. In each instance the excellent agreement between the actual displacement, as recorded by the condenser plates attached to the master oscillator (upper curve), shows plainly that this type of seismometer reproduces transients with a high degree of accuracy.

The high-frequency vibrations apparent as a broadening of the line at several points are due to vibration of the oscillograph suspension rather than to defects in the seismometer. The midpoint in the broad band may be considered the true location of the curve.

DIAPHRAGM-TYPE SEISMOMETER

Seismometers in which the steady mass is suspended from thin metal diaphragms are sometimes used. A diaphragm instrument using the condenser-plate system of recording was constructed according to the design shown in figure 18. (See also fig. 19.) The natural frequency of such an instrument is high. The instrument was placed on a solid table and the diaphragm jarred to give the record shown in figure 20. The beats are due to a slight difference in frequency between the two diaphragms, each vibrating at about 120 cycles per second.

⁹ Rockwell, E. H., *Vibrations Caused by Quarry Blasting and Their Effect on Structures*: Eng. and Contracting, vol. 66, no. 7, July 1927, p. 301.

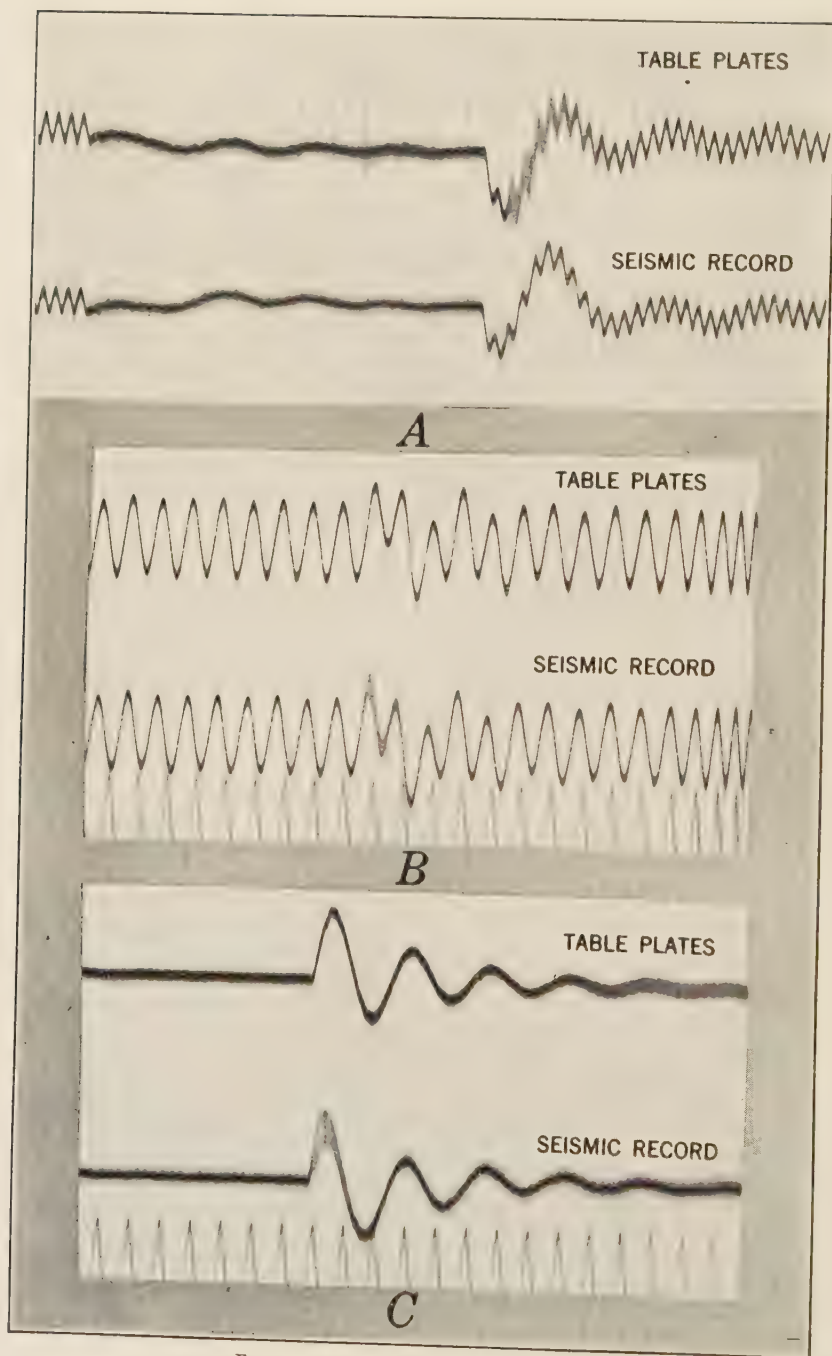


FIGURE 17.—Transient vertical vibrations.

Timoshenko¹⁰ shows that for a slightly damped seismic recorder the amplitude of the recorded curve is given approximately by the equation,

$$A = \frac{b}{p^2 - 1},$$

where b is a constant for the instrument, p is 2π times the natural frequency of the instrument, and m is 2π times the frequency of the

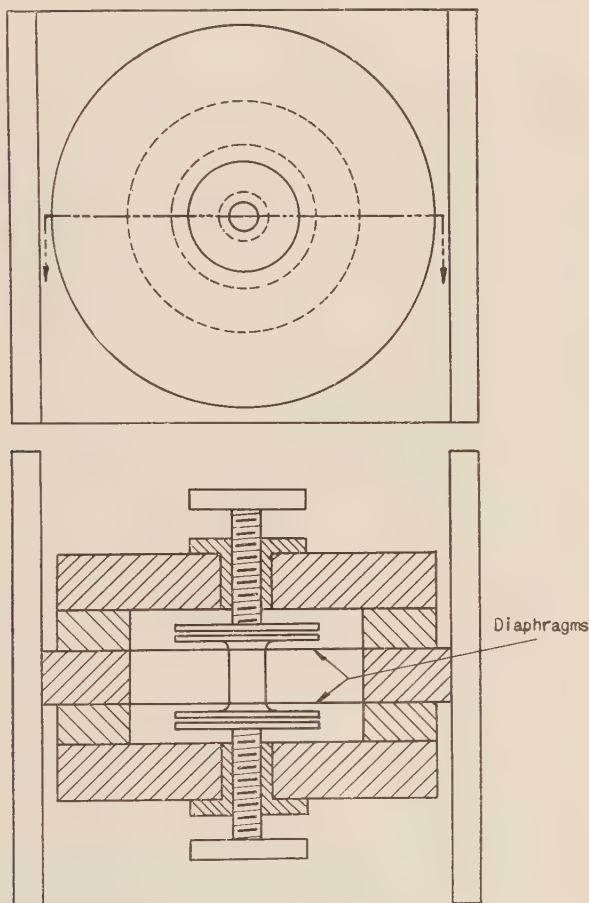


FIGURE 13. Diaphragm-type seismometer.

applied vibration. This equation shows that if the natural frequency of the instrument is very low compared to the frequency of the vibration being recorded $p \ll m$ and $A = -b$, approximately. The instrument will then record displacement with an amplitude independent of the frequency. The minus sign indicates that upward movement of the frame corresponds to downward movement of the middle plate, as is the case for the hinged-arm-type seismometer described in this paper.

¹⁰ Timoshenko, S., *Vibration Problems in Engineering*: Van Nostrand Co., 1928, p. 36.

In the diaphragm-type instrument used in this experiment the natural frequency was about 120 cycles, so that $p \gg m$. For this condi-

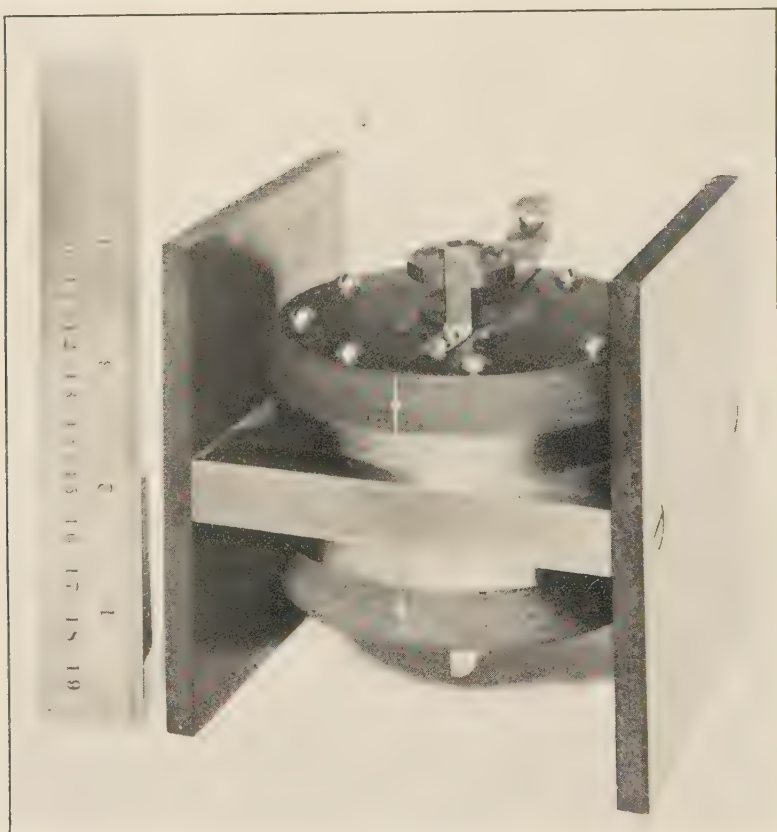


FIGURE 19.—Diaphragm-type seismometer.

tion the equation becomes approximately: $A = b \frac{m^2}{p^2}$. Since the sign of this quantity is plus, whereas the sign for the displacement was

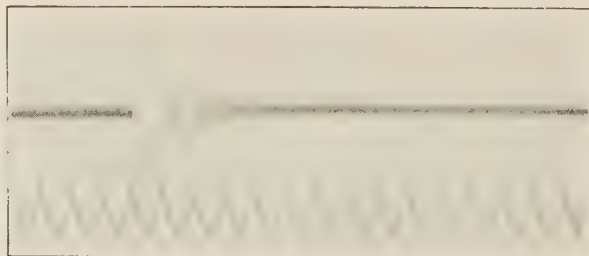


FIGURE 20.—Oscillograph record from diaphragm-type instrument, showing vibrations of diaphragm; diaphragm jarred; timing wave, 12.

minus, this instrument gives a curve 180° out of phase with the displacement, agreeing with the phenomenon of phase reversal previously discussed. Also, since p is a constant the curve has an amplitude proportional to m^2 , that is, proportional to the square of the frequency of

the applied vibration. Such a curve is a curve of acceleration, and the instrument is called an accelerometer.

This fact may be shown as follows:

Equation of displacement: $x = X \sin mt$, zero phase.

Equation of velocity: $\frac{dx}{dt} = mX \cos mt$, 90° phase angle.

Equation of acceleration: $\frac{d^2x}{dt^2} = -m^2X \sin mt$, 180° phase angle

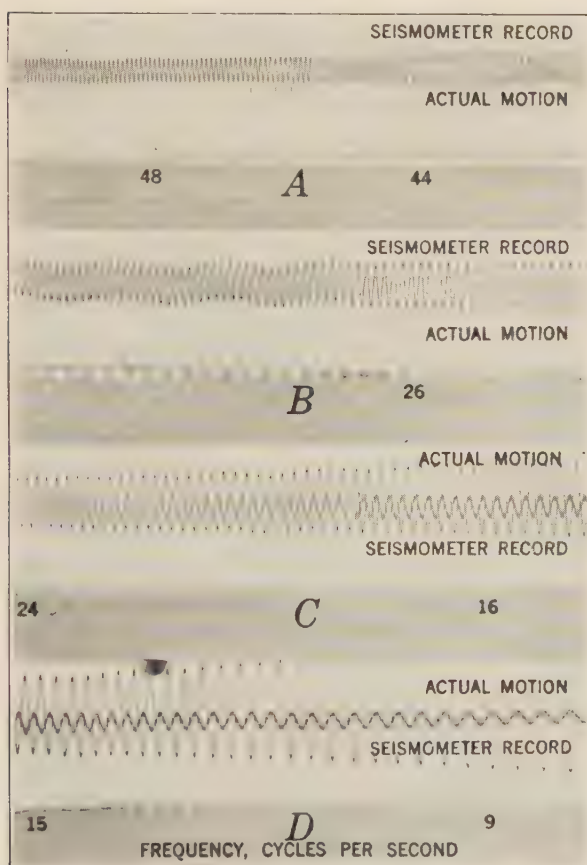


FIGURE 21.—Oscillograph records from diaphragm-type instrument with table-plate curves superimposed. (Decreasing frequency.)

Figure 21 shows curves taken with decreasing frequency. The displacement curve was taken by the condenser plates on the mechanical oscillator, and the instrument curve is shown with the same zero line. The heavy curve in A and B shows the actual motion, while in C and D the curve of actual motion appears lighter because of the greater amplitude. As the frequency of vibration decreases the amplitude of the oscillator increases because of the decreased mechanical impedance of the oscillator, but the amplitude of the curve registered by the instrument decreases because it is proportional to the square of the

frequency. The instrument curve is approximately 180° out of phase with the displacement, thus agreeing with the theory. Above 15 cycles per second the instrument shows high-frequency harmonics, which indicate that the diaphragm is vibrating at its own natural frequency. At 26, 44, and 48 cycles per second high-frequency vibrations of large amplitude conceal completely the record of acceleration of applied vibration. Figure 22, *B*, shows that at 8.5 cycles per second the instrument failed to show any response to the vibrations because of the low frequency, even though the amplitude was about 50 percent greater than in figure 22, *A*, where the curve registered by the instrument has considerable amplitude. Figure 22, *C*, shows the performance of this same instrument when a steady vibration of 35 cycles and a damped vibration of about 2 cycles are combined in the displacement curve. The instrument shows an acceleration curve for the higher frequency but no response to the low-frequency damped wave.

These curves show that an instrument of this type gives a greatly distorted wave at certain critical frequencies, so that the amplitude of

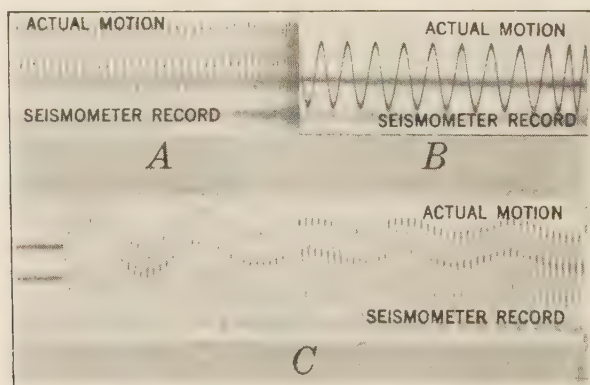


FIGURE 22.—Curves from diaphragm-type instrument and table-plate curves.

the acceleration cannot be determined near these frequencies. Stretching the diaphragm would raise its natural frequency and reduce this tendency but would greatly reduce the sensitivity. An instrument of this type would be poorly adapted for measuring displacement, as one having a low enough natural frequency would be very difficult to make. It must be concluded, therefore, that an instrument of this type is unsatisfactory for general use in vibration measurements at low frequencies where quantitative results are desired. The instrument has good sensitivity and may be used to advantage for detecting time of arrival of vibrations in certain geophysical explorations and for either horizontal or vertical vibrations, depending upon which side it is placed. It is very simple in construction, small, light, and very rugged.

CONCLUSIONS

Tests upon the mechanical oscillator indicate that a seismometer operating on the principle of change of capacitance between condenser plates has the necessary requirements for a portable instrument which will give a record of displacement-time curves. They also show that an instrument of this type can be adjusted quickly and easily for dif-

ferent sensitivities; after adjustment it can be calibrated accurately and quickly without any auxiliary apparatus.

Theoretical analysis of the motion at the center of oscillation of the arm shows that it is possible to design an instrument of small mass which will have a steady point sufficiently stationary over a given range of frequency to give satisfactory results. Experiments with the mechanical oscillator confirm the results of the mathematical study and prove that a properly designed and adjusted seismometer can record transients and complex wave shapes with good accuracy.

In testing instruments having different characteristics the author found that a seismometer produces greatly distorted records when operated near its natural frequency. Phase displacement and sensitivity increase markedly as the resonant frequency of the instrument is approached. When complex waves are being recorded several frequencies are present. The record shows serious distortion if the fundamental and all the harmonic frequencies are not recorded with negligible phase displacement and equal sensitivity. Consequently, seismometers designed to operate near their natural frequency to give high sensitivity or shorten the duration of transient effects cannot produce an accurate record of displacement.

Nearly critical damping is an essential feature of a seismometer. Care in design makes possible an instrument having a low natural frequency and small mass, so that critical damping by magnetic means is not difficult.

Means of supporting the mass of a vertical-component seismometer against the force of gravity is a serious problem. Use of helical springs for this purpose introduces a new source of error due to the tendency of the spring to vibrate transversely. Springs must be made short to raise the natural frequency for transverse vibrations well above the highest value to be recorded. Short springs have a large change in tension for small change in length and must be mounted so that the change in pull with rotation of the arm is nearly compensated for; otherwise, the natural period of the arm will be too short. The spring can be attached so that the moment of the force it exerts is nearly constant. This type of construction seems to be the most satisfactory method yet devised for supporting the weight of the arm.

An instrument in which the mass was supported by a diaphragm was tested and found to have a high natural frequency. It behaved as an accelerometer; the distortion at certain critical frequencies was marked. As the restoring force is inherently great in this type of construction the design of a diaphragm instrument with a natural frequency low enough to record displacement at low frequency obviously would be difficult.

The sensitivity of a compensated-spring-type vertical-component seismometer (fig. 14) is constant at 2.5 to 22 cycles per second and almost constant to 65 cycles per second. Horizontal-component seismometers have a high upper limit, well beyond the highest frequency occurring in ground vibrations. If a different range of frequencies is desired the design theory developed in this paper may be used to predict the performance of any proposed design, at any frequency, before construction. The response of the seismometer to transient vibrations and complicated wave shapes shows that it will record the actual curve of displacement for impulses and rapidly occurring transients.

APPENDIX

DERIVATION OF EQUATION OF MOTION OF CENTER OF OSCILLATION

Several investigators¹¹ have given a complete analysis of the theory of the seismometer. In these solutions the equation has usually been written in terms of the angle of rotation of the pendulum, and the dimensions have been units of acceleration or force. Where sinusoidal vibrations have been assumed the phase angle has been that between radians $\times$ angular rotation and applied acceleration or force. In this paper it was desirable to determine the magnitude of displacement of the center of oscillation with respect to a fixed point in space in terms of the design constants and to find the phase angle between this displacement and that of the frame. The author realizes that the solution given here is only approximate, but the results obtained from tests of the seismometer on the mechanical oscillator indicate that the conclusions are dependable enough for practical application.

Figure 23 shows that when the frame moves distance x_1 , the center of oscillation of the hinged arm moves distance x_2 and the arm rotates

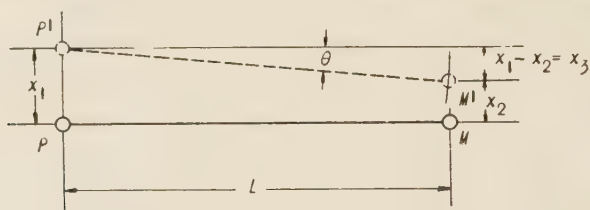


FIGURE 23. -Diagram showing rotation of seismometer pendulum.

about the axis of the hinge an angle of θ . If L is the effective length of the arm or distance from the hinge to the center of oscillation (M),

$$\sin \theta = \frac{x_1 - x_2}{L}.$$

As θ is always very small $\sin \theta = \theta$, approximately.

A restoring force tends to hold θ equal to zero, and a damping force opposes change of angle θ . Each of these forces opposes rotation of the arm, thus tending to make $x_2 = x_1$. On the other hand, mass M , due to its inertia, tends to remain at a fixed point in space and so to make x_2 equal to zero. The friction of the hinge and the fanning of the outside plates are not considered separately but are assumed to add to the effect of the magnetic damping. This assumption is not strictly correct, as the friction of hinge and air is not directly proportional to velocity, but the damping due to friction is so small compared to magnetic damping that the error is negligible. This statement can be proved experimentally by removing the damping magnets and noting how small decrement of the arm was due to friction alone.

If F_M is the force due to the inertia of the mass, F_R the force due to the restoring torque, and F_H the force due to the damping action the relation

$$F_M = F_R + F_H \text{ holds.} \quad (1)$$

¹¹ Reid, H. F., The California Earthquake: Carnegie Inst., 1910, vol. 2, p. 148. This report gives several references to earlier publications on the mathematical theory of seismographs.

The restoring force tending to hold displacement angle θ equal to zero is the component of force K , which acts at right angles to the arm. This component of K is equal to $K \sin \theta$, so that

$$F_R = K \sin \theta = \frac{K}{L} (x_1 - x_2). \quad (2)$$

For the horizontal-component seismometer $K = Mg \sin \alpha$ (see fig. 1). The value of K for the vertical-component seismometer will be developed later.

Damping force F_H is produced by the damping vane moving through the field of the permanent magnet. As the magnetic flux is constant the force opposing motion will be proportional to the velocity of the plate with respect to the poles. The poles are fixed to the same frame as the hinge, so the relative motion of the vane through the field is proportional to displacement angle θ and the velocity of motion is proportional to $\frac{d\theta}{dt}$. This force acts at a longer radius than forces F_R and F_M ; the actual force should therefore be multiplied by the ratio of the two lengths. As this ratio is fixed by the design it may be included in the constant of the damping system, and one constant of proportionality (H) may be used. Then

$$F_H = H \frac{d\theta}{dt} = \frac{H}{L} \frac{d(x_1 - x_2)}{dt} = \frac{H}{L} \left(\frac{dx_1}{dt} - \frac{dx_2}{dt} \right). \quad (3)$$

The force due to inertia of mass M is

$$F_M = \text{mass multiplied by acceleration} = M \frac{d^2 x_2}{dt^2}. \quad (4)$$

Substituting the values of these three forces in equation (1),

$$M \frac{d^2 x_2}{dt^2} = \frac{K}{L} (x_1 - x_2) + \frac{H}{L} \left(\frac{dx_1}{dt} - \frac{dx_2}{dt} \right). \quad (5)$$

If the vibration of the instrument is assumed to be sinusoidal and X_1 represents the maximum value of the amplitude and ω equals 2π multiplied by the frequency of vibration, then,

$$x_1 = X_1 \sin \omega t \text{ and } \frac{dx_1}{dt} = \omega X_1 \cos \omega t.$$

Substituting these values in equation (5),

$$M \frac{d^2 x_2}{dt^2} = \frac{K}{L} X_1 \sin \omega t - \frac{K}{L} x_2 + \frac{H\omega}{L} X_1 \cos t - \frac{H}{L} \frac{dx_2}{dt}. \quad (6)$$

Dividing through by M , making $Z = \sqrt{(H\omega)^2 + K^2}$ and $\beta = \tan^{-1} \frac{H\omega}{K}$, and using operator $p = \frac{d}{dt}$, the equation becomes

$$\left(p^2 + \frac{H}{LM} p + \frac{K}{LM} \right) x_2 = \frac{Z}{LM} X_1 \sin (\omega t + \beta). \quad (7)$$

To obtain the general solution let the right member equal zero. Then,

$$p = -\frac{H}{2LM} \pm \sqrt{\left(\frac{H}{2LM}\right)^2 - \frac{K}{LM}}$$

For an instrument with small friction and no artificial damping $\frac{K}{LM} > \left(\frac{H}{2LM}\right)^2$. In this case the radical is imaginary and the arm tends to oscillate at a frequency of

$$\nu = \frac{1}{2\pi} \sqrt{\frac{K}{LM} - \frac{H^2}{4L^2M^2}} \quad (8)$$

Damping factor H depends upon the construction and may be so adjusted that $H = 2\sqrt{KLM}$, the radical becomes equal to zero, and the instrument is critically damped. If the instrument is underdamped the middle plate will continue to swing at a frequency of ν , given by equation (8), if displaced, until the energy stored in the system by the displacement is dissipated by the damping. An example of this swinging in an underdamped seismometer is shown in figure 2. For satisfactory operation the damping should be critical or nearly so to avoid this swinging.¹² It should not greatly exceed the critical value, as the damping opposes the inertia effect and tends to pull the middle plate along with the frame. As critical damping is necessary for satisfactory operation the solution for this condition is the only one of practical interest.¹³ In the solution from this point $2\sqrt{KLM}$ will be substituted for H , and only the case of critical damping will be studied.

If the damping is assumed to be zero the value of ν would check with the value for the natural frequency of the arm, as given on page 4,

$$\nu = \frac{1}{2\pi} \sqrt{\frac{g \sin \alpha}{L}}$$

If $H=0$ in equation (8),

$$\nu = \frac{1}{2\pi} \sqrt{\frac{K}{LM}},$$

but $K = Mg \sin \alpha$, so that

$$\frac{K}{LM} = \frac{g \sin \alpha}{L}$$

and the two equations are identical for this condition.

The general solution for the condition of critical damping is,

$$x_2' = A\epsilon^{-at} + Bt\epsilon^{-at}, \quad (9)$$

where

$$a = 2\frac{H}{LM} = \sqrt{\frac{K}{LM}}, \text{ and } A \text{ and } B \text{ are constants of integration.}$$

Equation (9) is the transient term of the solution.

¹² Anderson, J. A., and Wood, H. O., Description and Theory of the Torsion Seismometer: Bull. Seismol. Soc. America, vol. 15, no. 1, p. 51.

¹³ Solutions for the cases of underdamping and overdamping are given in the paper by Anderson and Wood, pp. 48-50.

The particular integral or steady-state term can be found by replacing p by $j\omega$. Substituting and simplifying,

$$x_2'' = \frac{1}{-\omega^2 + \frac{K}{LM} - j\frac{H\omega}{LM}} \frac{Z}{LM} X_1 \sin(\omega t + \beta) \quad (10)$$

$$= \frac{Z}{LM\omega^2 + K} X_1 \sin(\omega t + \delta), \quad (11)$$

where $\delta = \beta - \gamma$

$$\text{and } \gamma = \tan^{-1} \frac{2\omega\sqrt{KLM}}{K - \omega^2 LM}.$$

Coefficient $\frac{ZX_1}{LM\omega^2 + K}$ depends upon the design constants of the instrument and the amplitude and frequency of the applied vibration. To simplify let

$$C = \frac{ZX_1}{LM\omega^2 + K}.$$

Then

$$x_2'' = C \sin(\omega t + \delta) + A\epsilon^{-at} + Bt\epsilon^{-at}, \quad (12)$$

The complete solution for x_2 is the sum of the general solution and the particular integral or the sum of the transient and steady-state terms,

$$x_2 = x_2' + x_2'' = C \sin(\omega t + \delta) + A\epsilon^{-at} + Bt\epsilon^{-at}. \quad (13)$$

To evaluate the constants A and B let $t=0$. Let $(x_2)_0$ represent the value of x_2 or the displacement at the center of oscillation when $t=0$, and let $(\dot{x}_2)_0 = \frac{dx_2}{dt}$, the velocity of motion at the center of oscillation when $t=0$.

Then,

$$(x_2)_0 = C \sin \delta + A \text{ and } A = (x_2)_0 - C \sin \delta. \quad (14)$$

$$\frac{dx_2}{dt} = \omega C \cos(\omega t + \delta) - aA\epsilon^{-at} + B\epsilon^{-at} - aBt\epsilon^{-at}.$$

If $t=0$,

$$\begin{aligned} (\dot{x}_2)_0 &= \omega C \cos \delta - aA + B \\ &= \omega C \cos \delta - a(x_2)_0 + aC \sin \delta + B. \end{aligned}$$

Hence,

$$B = (\dot{x}_2)_0 - C(\omega \cos \delta - a \sin \delta) + a(x_2)_0. \quad (15)$$

Substituting values for A and B and collecting terms,

$$x_2 = C\{\sin(\omega t + \delta) - \epsilon^{-at}[\sin \delta + t(\omega \cos \delta + a \sin \delta)]\} + \epsilon^{-at}\{(x_2)_0 + t[(\dot{x}_2)_0 + a(x_2)_0]\}. \quad (16)$$

Since seismic disturbances consist of a series of rapidly occurring transients no simplification of this equation by equating $(x_2)_0$ or $(\dot{x}_2)_0$ to zero has any significance. The assumption that the instrument is at a certain position when the disturbance begins or that the disturbance has such a wave shape that its initial velocity is zero does not agree with actual working conditions.¹⁴ By careful design, however, the value of C may be made very small for all frequencies within the

¹⁴ Galitzin, F. B., Über seismometrische Beobachtungen: Compt. rend., Acad. Imp. Soc. Commission Sismique Permanente, vol. 1, p. 104.

useful range of the seismometer. When the instrument is so designed x_2 will always have a negligible value after the transient terms have disappeared.

Since e^{-at} , $\sin(\omega t + \delta)$, $\sin \delta$, and $\cos \delta$ can never exceed unity their product with a small value of C will also be small. If these terms are neglected

$$x_2 = e^{-at} \{ (x_2)_0 + t[(\dot{x}_2)_0 + a(x_2)_0] \}. \quad (17)$$

Initial displacement of the center of oscillation $(x_2)_0$ and its initial velocity $(\dot{x}_2)_0$ at the instant the disturbance begins cannot be determined. Either may be zero or may have any value up to its maximum. Even the maximum values of displacement and velocity of the center of oscillation are probably very small in a seismometer having a negligible value for C ; if so the seismometer will record accurately during the transient period. This probability cannot be tested mathematically. The only proof that the seismometer will perform satisfactorily is obtained by testing it on the mechanical oscillator and comparing the actual curve of motion recorded by the condenser plates on the table with the record of the instrument while a complicated transient is produced.

Obviously, the important factor in good design is to obtain a small value for coefficient C within the range of frequencies for which the instrument is to be used, as this value controls both transient and steady-state conditions. The instrument can be so designed that the transient will die out quickly by causing a to have a large value.¹⁵ This method of reducing the error due to transients is not feasible for two reasons. First, the initial value of the transient is important even though the decay is very rapid, and large values for a tend to increase the initial value of the transient since a occurs as a factor in several positive terms. Second, design factors that increase a also increase C , as may be readily seen.

$$a = \sqrt{\frac{K}{LM}}; \quad C = X_1 \frac{Z}{LM\omega^2 + K} = \frac{\sqrt{4KLM\omega^2 + K^2}}{LM\omega^2 + K} X_1.$$

As $LM\omega^2$ becomes very large compared to K the value of C approaches zero. Since ω depends upon the frequency of the vibrations to be measured it cannot be controlled, and $LM\omega^2$ must be made large compared to K by making LM large compared to K . Therefore, the conditions that reduce the value of C must also reduce the value of a . The better practice is to design an instrument having a transient that continues for a number of cycles but is negligible even for its initial value rather than to attempt elimination of transient effects by reducing their duration.

For the steady-state condition x_2 reaches the maximum value when $\sin(\omega t + \delta) = 1$. If X_2 is the maximum value of x_2 the ratio

$$\frac{X_2}{X_1} = \frac{\sqrt{4KLM\omega^2 + K^2}}{LM\omega^2 + K}.$$

As the frequency of vibration approaches zero the motion of the middle plate must approach that of the frame, or $\frac{X_2}{X_1}$ must approach unity. This condition is satisfied when $\omega = 0$ in the equation for the ratio.

¹⁵ Anderson, J. A., and Wood, H. O., Description and Theory of the Torsion Seismometer: Bull. Seismol. Soc. America, vol. 15, no. 1, p. 53.

Also, it is apparent that as the frequency becomes very high the value of the ratio should approach zero. If $\omega = \infty$ the value of the ratio is zero.

To find the maximum value of the ratio the first derivative with respect to ω may be equated to zero. The result of this process shows that for this condition $\omega^2 = \frac{K}{4ML} (-1 \pm 3)$. The negative value must be discarded, as it gives an imaginary value for the frequency. Thus $\omega = \sqrt{\frac{K}{2ML}}$, the condition for maximum movement of the steady mass.

The value of the ratio found by substituting this value for ω in the equation shows that the maximum value of the ratio is $\left(\frac{X_2}{X_1}\right)_{\max.} = \frac{\sqrt{3}K^2}{1.5K} = 1.155$. This value, however, applies only to a critically damped pendulum. For an underdamped instrument the maximum value of this ratio becomes very great, approaching infinity at resonance for zero damping.

RELATIVE MOTION OF MIDDLE PLATE

Evidently, in studying the effect of movement of the middle plate upon the record produced by the seismometer both the phase angle and amplitude of vibration must be computed. To simplify this analysis (1) the middle plate is assumed to remain parallel to the outer plates at all times—an assumption not strictly true—and (2) the center of oscillation of the arm is assumed to coincide with the center line of the condenser plate. The second assumption may or may not be correct, depending upon the distribution of the mass in the hinged arm. The errors due to the assumptions are discussed under Error Due to Movement of Middle Plate (p. 7).

Let $x_1 = X_1 \sin \omega t$ = the equation of motion of the frame.

$x_2 = X_2 \sin (\omega t + \delta)$ = the equation of motion of the middle plate.

$x_3 = X_3 \sin (\omega t + \phi)$ = the equation of recorded wave.

If the transient terms are neglected the equation for x_2 in terms of the constants of the instrument and the frequency of vibration is $X_2 = \frac{Z}{LM\omega^2 + K} X_1$; $\frac{X_2}{X_1}$ may therefore be found for any frequency if the constants of the instrument are known.

Since $x_1 = 0$ when $t = 0$, ϕ is the error in phase of the recorded curve or the phase angle between the actual vibration of the seismometer and the record obtained. Deflection of the oscillograph depends upon change of capacitance between the plates, so that the maximum of the recorded curve will occur when the middle plate is closest to one outer plate and farthest from the other. Since x_1 and x_2 are measured from parallel axes, the distance between the upper and middle plates is $A + x_3 = A + x_1 - x_2$. Distance A is constant for any given adjustment and may be dropped, as it affects only the zero position and sensitivity, but not the value of x_3 . Therefore, $x_3 = x_1 - x_2$. (See fig. 4.)

Then,

$$x_3 = X_3 \sin (\omega t + \phi) = x_1 - x_2. \quad (18)$$

Let X_1 , X_2 , and X_3 be vectors generating the sine waves x_1 , x_2 , and x_3 and assume that X_1 is the reference vector.

Then,

$$X_1 = X_1 + j0. \quad (19)$$

$$X_2 = X_2 \cos \delta + jX_2 \sin \delta. \quad (20)$$

$$X_3 = X_1 - X_2 \cos \delta - jX_2 \sin \delta = X_3 \cos \phi - jX_3 \sin \phi. \quad (21)$$

Hence,

$$\tan \phi = \frac{-X_2 \sin \delta}{X_1 - X_2 \cos \delta} = \frac{-\frac{X_2}{X_1} \sin \delta}{1 - \frac{X_2}{X_1} \cos \delta}. \quad (22)$$

Also,

$$\begin{aligned} |X_3| &= \sqrt{(X_1 - X_2 \cos \delta)^2 + (X_2 \sin \delta)^2} \\ &= \sqrt{X_1^2 - 2X_1X_2 \cos \delta + X_2^2 \cos^2 \delta + X_2^2 \sin^2 \delta} \\ &= \sqrt{X_1^2 - 2X_1X_2 \cos \delta + X_2^2} \end{aligned} \quad (23)$$

$$\frac{X_3}{X_1} = \sqrt{1 - 2\frac{X_2}{X_1} \cos \delta + \left(\frac{X_2}{X_1}\right)^2}. \quad (24)$$

From equations (22) and (24) ϕ and X_3 can be computed. For perfect accuracy $\phi=0$ and $X_3=X_1$. This condition is approached as X_2 approaches zero.

By use of trigonometric relations equation (24) may be simplified into a form more convenient for use in computation of numerical values.

$$\tan \beta = 2\omega \sqrt{\frac{LM}{K}} \text{ from equation (7).}$$

$$\delta = \beta - \gamma \text{ and } \tan \gamma = \frac{2\omega \sqrt{KLM}}{K - \omega^2 LM} \text{ from equation (11).}$$

$$\cos \delta = \cos (\beta - \gamma) = \cos \beta \cos \gamma + \sin \beta \sin \gamma.$$

$$\cos \beta = \frac{1}{\sqrt{1 + \tan^2 \beta}} = \frac{1}{\sqrt{1 + \frac{4\omega^2 LM}{K}}} = \sqrt{\frac{K}{K + 4\omega^2 LM}}.$$

$$\cos \gamma = \frac{1}{\sqrt{1 + \tan^2 \gamma}} = \frac{1}{\sqrt{1 + \frac{4\omega^2 KLM}{(K - \omega^2 LM)^2}}} = \frac{K - \omega^2 LM}{K + \omega^2 LM}.$$

$$\sin \beta = \tan \beta \cos \beta = 2\omega \sqrt{\frac{LM}{K + 4\omega^2 LM}}.$$

$$\sin \gamma = \tan \gamma \cos \gamma = \left(\frac{2\omega \sqrt{KLM}}{K - \omega^2 LM} \right) \left(\frac{K - \omega^2 LM}{K + \omega^2 LM} \right) = \frac{2\omega \sqrt{KLM}}{K + \omega^2 LM}.$$

$$\begin{aligned} \cos \delta &= \frac{K - \omega^2 LM}{K + \omega^2 LM} \sqrt{\frac{K}{K + 4\omega^2 LM}} + \frac{4\omega^2 LM \sqrt{K}}{(K + \omega^2 LM) \sqrt{K + 4\omega^2 LM}} \\ &= \frac{K + 3\omega^2 LM}{K + \omega^2 LM} \sqrt{\frac{K}{K + 4\omega^2 LM}} \end{aligned}$$

$$\frac{X_2}{X_1} = \frac{\sqrt{4KLM\omega^2 + K^2}}{LM\omega^2 + K} \text{ from equation on page 34.}$$

Substituting these two values in equation (24),

$$\frac{X_3}{X_1} = \sqrt{1 - 2\sqrt{\frac{4KLM\omega^2 + K^2(K + 3\omega^2LM)}{(LM\omega^2 + K)^2}}} \sqrt{\frac{K}{K + 4\omega^2LM} + \frac{4KLM\omega^2 + K^2}{(LM\omega^2 + K)^2}}.$$

This reduces to

$$\frac{X_3}{X_1} = \frac{1}{1 + \frac{K}{LM\omega^2}}. \quad (26)$$

Similarly, equation (22) may be reduced, using $\sin \delta = \sin (\beta - \gamma) = \sin \beta \cos \gamma - \cos \beta \sin \gamma$ in addition to the values given above. The result of this simplification is,

$$\tan \phi = -\frac{2\omega\sqrt{KLM}}{K - \omega^2LM}. \quad (27)$$

This expression is the same as $-\tan \gamma$. Table 1 shows that $180^\circ - \gamma = \phi$. Therefore their tangents must be equal numerically but are opposite in sign.

When these results are transformed into the same nomenclature they agree with the solution of Anderson and Wood,¹⁶ although these investigators used a more direct method of solution. The method given here has the advantage of showing the displacement of the center of mass in terms of the design constants and the applied vibration in terms of its displacement.

For critical damping Anderson and Wood get the value $\frac{\mathcal{A}}{\mathcal{M}}$ for $\theta_{\max.}$,

in which $\mathcal{A}$ is the applied angular acceleration and $\mathcal{M} = 4\pi^2 \left(\frac{1}{T_o^2} + \frac{1}{T_e^2} \right)$.

Converting these quantities to the symbols used in this paper,

$$\mathcal{A} \text{ corresponds to } \frac{-\omega^2 X_1}{L},$$

$$\frac{1}{T_o^2} \text{ corresponds to } \frac{1}{4\pi^2} \frac{K}{LM}, \text{ and } \frac{1}{T_e^2} \text{ to } \frac{\omega^2}{4\pi^2}.$$

Substituting these values and reducing, $\mathcal{M} = \frac{K}{LM} + \omega^2$.

Hence,

$$\theta_{\max.} = \frac{\mathcal{A}}{\mathcal{M}} = \frac{-\omega^2 X_1}{L \left(\frac{K}{LM} + \omega^2 \right)} = \frac{-\omega^2 M X_1}{K + LM\omega^2},$$

$$X_3 = \theta_{\max.} L = \frac{-X_1 \frac{K}{LM}}{1 + \frac{LM\omega^2}{K}}.$$

Or,

$$\frac{X_3}{X_1} = \frac{-1}{1 + \frac{K}{LM\omega^2}}.$$

¹⁶ Anderson, J. A., and Wood, H. O., Description and Theory of the Torsion Seismometer: Bull. Seismol. Soc. America, vol. 15, no. 1, p. 152.

This agrees with equation (26) except for sign, and as equation (26) is only the absolute value this difference has no significance. The minus sign indicates that when the frame moves up distance X_1 is measured upward and X_3 is measured down from a horizontal line. (See fig. 23.)

Anderson and Wood give for phase angle between applied angular acceleration and angle of rotation,

$$\delta = \tan^{-1} \frac{2T_a T_r}{T_r^2 - T_a^2}.$$

Substituting the corresponding symbols from this paper, $\tan \delta = \frac{2\omega \sqrt{KLM}}{K - \omega^2 LM}$ which agrees with $\tan \gamma$ in this paper. As acceleration is 180° out of phase with displacement, while the angle of rotation is in phase with displacement, phase angle δ from their paper should be 180° out of phase with ϕ of this paper and so would agree with γ .

PRACTICAL APPLICATION OF THEORY

The use of equations (26) and (27) for computation of error in amplitude and phase for a seismometer are shown in table 1; the results are plotted in figure 5. The constants for one instrument were found from measurement to be as follows: $M=443$ grams; $L=6$ cm; and $\alpha=5^\circ$.

These values give the following results: $K-Mg \sin \alpha=37,900$ dynes; $LM=2,658$ $\sqrt{KLM}=10,025$.

TABLE 1.—*Computation of error in amplitude and phase at different frequencies*

Cycles	ω	2ω	ω^2	$LM\omega^2$	$\frac{K}{LM\omega^2}$	$1 + \frac{K}{LM\omega^2}$
1	6.3	12.6	39	0.105×10^6	0.3620	1.3620
3	18.8	37.6	355	$.944 \times 10^6$	.0401	1.0401
5	31.4	62.8	988	2.63×10^6	.0144	1.0144
10	62.8	125.6	3,940	10.48×10^6	.0036	1.0036
20	125.6	251.2	15,650	41.72×10^6	.0009	1.0009

$\frac{X_3}{X_1}$	Error, percent	$2\omega \sqrt{KLM}$	$K - LM\omega^2$	Tan γ	γ	ϕ
0.735	26.5	126,300	-66,900	-1.888	122.1	57.9
.961	3.9	378,900	-906,100	-.418	157.3	22.7
.985	1.5	631,500	-2.59 $\times 10^6$	-.243	166.3	13.7
.995	.5	1,263,000	-10.45 $\times 10^6$	-.121	173.1	6.9
1.000	0	2,526,000	-41.72 $\times 10^6$	-.061	176.5	3.5

Table 2 shows complete computations for an instrument, the critical frequency of which is within the range considered. Some of the values in the table are plotted in figure 24.

Let $M=25$ grams, $L=4$ cm, and $K=60,000$ dynes.

Then the maximum vibration of the steady mass will occur at a frequency of $\frac{1}{2\pi}\sqrt{\frac{K}{2ML}} = \frac{1}{2\pi}\sqrt{\frac{60,000}{2 \times 25 \times 4}} = 2.76$ cycles (fig. 24). The

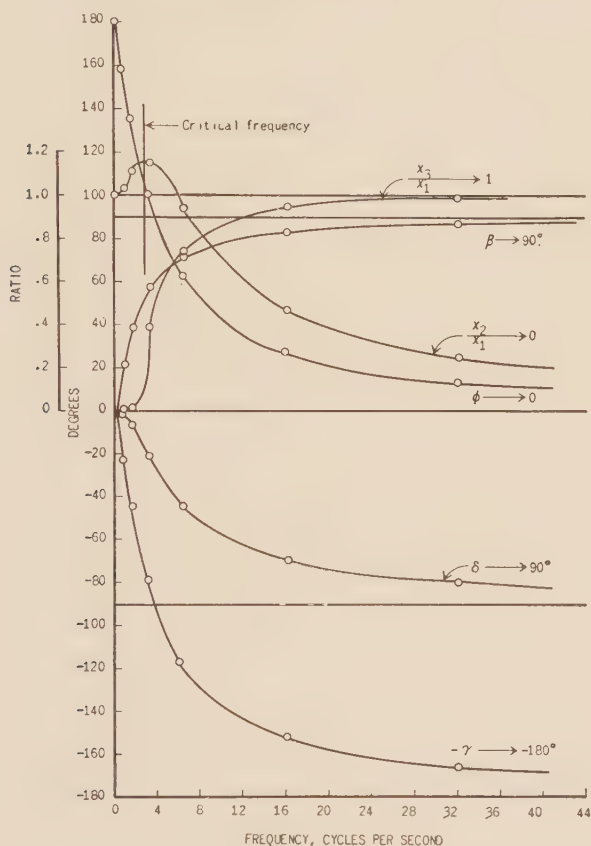


FIGURE 24.—Curves plotted from computations, table 2.

ratio $\frac{X_2}{X_1}$ reaches its maximum (1.155) at this frequency, and the phase angle ϕ is about 90° at that point. This agrees with the oscillograph record, figure 6.

Table 2 shows that $\tan \gamma = -\tan \phi$ and that $\phi = 180^\circ - \gamma$.

$LM=4 \times 25=100$; $K=6 \times 10^4$; $K^2=36 \times 10^8$; $KLM=6 \times 10^6$; $4KLM=24 \times 10^6$;

$$\sqrt{KLM}=2.45 \times 10^3; \text{ and } \sqrt{\frac{LM}{K}}=0.04082.$$

TABLE 2.—*Computations for obtaining phase angle ϕ and ratio $\frac{X_3}{X_1}$*

Cycles	ω	ω^2	2ω	$2\omega\sqrt{\frac{LM}{K}}$	$\beta, ^\circ$
0.8	5	25	10	0.4082	22.24
1.6	10	100	20	.8164	39.11
3.2	20	400	40	1.6328	58.51
6.4	40	1,600	80	3.2656	72.95
16.0	100	10,000	200	8.164	83.02
32.0	200	40,000	400	16.328	86.50

$2\omega\sqrt{KLM}$	ω^2LM	$K-\omega^2LM$	$\tan \gamma$	$\gamma, ^\circ$	$\delta, ^\circ$
2.45×10^4	0.25×10^4	5.75×10^4	0.426	23.08	-0.84
4.9×10^4	1.0×10^4	5.0×10^4	.98	44.42	-5.31
9.8×10^4	4.0×10^4	2.0×10^4	4.90	78.5	-19.95
19.6×10^4	16.0×10^4	-1.0×10^5	-1.96	117.0	-44.05
49.0×10^4	1.0×10^5	-9.4×10^5	-.52	152.5	-69.48
98.0×10^4	4.0×10^5	-39.4×10^5	-.25	166.0	-80.50

$\sin \delta$	$\cos \delta$	$4\omega^2KLM$	Z^3	Z	ω^2LM+K
-0.01462	0.9999	6×10^5	42×10^8	6.48×10^4	6.25×10^4
-.926	.9957	24×10^5	60×10^8	7.75×10^4	7.0×10^4
-.3420	.941	96×10^5	132×10^8	11.5×10^4	1.0×10^4
-.6953	.719	384×10^5	420×10^8	20.5×10^4	2.2×10^5
-.936	.350	24×10^{10}	24.4×10^{10}	4.94×10^5	1.1×10^5
-.986	.165	96×10^{10}	96.4×10^{10}	9.82×10^5	4.1×10^5

$\frac{X_2}{X_1}$	$\left(\frac{X_2}{X_1}\right)^2$	$\frac{X_2}{X_1} \sin \delta$	$\frac{X_2}{X_1} \cos \delta$	$1 - \frac{X_2}{X_1} \cos \delta$	$\tan \phi$
1.036	1.07	0.01516	1.036	-0.036	-0.421
1.106	1.22	.1025	1.100	-.100	-.98
1.150	1.32	.393	1.081	-.081	-4.90
.933	.87	.649	.670	.330	1.96
.462	.214	.432	.162	.838	.52
.242	.059	.239	.040	.960	.25

$\phi, ^\circ$	$1 + \left(\frac{X_2}{X_1}\right)^2$	$2\frac{X_2}{X_1} \cos \delta$	$\left(\frac{X_3}{X_1}\right)^2$	$\frac{X_3}{X_1}$
157.1	2.07	2.07	0.000	0.000
135.6	2.22	2.20	.020	.141
101.5	2.32	2.162	.158	.397
63.0	1.87	1.340	.530	.728
27.5	1.214	.324	.890	.943
14.0	1.059	.080	.979	.988

THEORY OF DIRECT SPRING SUPPORT

A hinged arm directly supported by a helical spring acting at right angles to the arm satisfies the following equation:

$$F = M \frac{d^2x}{dt^2} + H \frac{dx}{dt} + kx,$$

F =force producing displacement,

M =mass of arm,

x =instantaneous displacement,

H =damping constant,

k =spring constant (force per unit length).

The solution of this equation shows that if $\frac{k}{M} > \frac{H^2}{4M^2}$ the arm will

tend to oscillate at a frequency of $\frac{1}{2\pi} \sqrt{\frac{k}{M} - \frac{H^2}{4M^2}}$. If the damping is

negligible the natural frequency of oscillation will be $\frac{1}{2\pi} \sqrt{\frac{k}{M}}$.

As shown in the discussion on the general theory of a seismometer, the natural frequency of vibration when undamped must be well below the frequency range for which the instrument is to be used. To keep the value of the natural frequency small k must be small or M large. The value of k for a helical spring is given by the equation,

$$k = \frac{F}{x} = \frac{Gd}{N} \left(\frac{d}{4R} \right)^3 \quad ^{17}$$

F =force applied to spring, pounds;

x =extension of spring due to F , inches;

G =modulus of torsional elasticity (for steel $G=12 \times 10^6$);

d =diameter of spring wire, inch;

R =mean radius of helix, inches;

N =number of turns in helix.

To make k small d must be made small and R or N large. If d is made small or R large the safe load which the spring can support is reduced so that for a given required pull there is a limit to these values. After this limit has been reached k may be reduced further only by increasing N , thus making the spring long.

The formula for safe loading of a helical spring is,

$$W = \frac{\pi}{16} \frac{Sd^3}{R} \quad ^{18}$$

W =safe load, pounds;

S =constant (for steel $S=60,000$);

d =diameter of spring wire, inch;

R =mean radius of helix, inches.

Computations have been made to show the effect of changes in spring constants on operation of the instrument.

Assume that required pull = 240 grams and diameter of spring wire = 0.014 inch. Find R for safe load.

$$R = \frac{\pi}{16} \frac{Sd^3}{W} = \frac{3.14 \times 60,000 \times (0.014)^3}{16 \times 240 \times 2.205 \times 10^{-3}} = 0.061 \text{ inch.}$$

Assume $N=50$, number of turns in spring.

$$k = \frac{Gd}{N} \left(\frac{d}{4R} \right)^3 = \frac{12 \times 10^6 \times 0.014}{50} \left(\frac{0.014}{4 \times 0.061} \right)^3 = 0.669.$$

0.669 pound per inch = 117,000 dynes per centimeter. Extension for pull of 240 grams = $\frac{240 \times 980}{117,000} = 2.01$ cm. Normal length of close-wound spring = $50 \times 0.014 \times 2.54 = 1.778$ cm. Length when supporting 240 grams = $2.01 + 1.78 = 3.79$ cm. Natural frequency when undamped,

$$n = \frac{1}{2\pi} \sqrt{\frac{k}{M}} = \frac{1}{6.28} \sqrt{\frac{117,000}{240}} = 3.52 \text{ cycles per second.}$$

¹⁷ Cloud, J. W., Trans. Am. Soc. Mech. Eng., vol. 173.

¹⁸ Cloud, J. W., work cited.

As a check on this computation,

$$n = \frac{1}{2\pi} \sqrt{\frac{g}{x}} = \frac{1}{6.28} \sqrt{\frac{980}{2.01}} = 3.52 \text{ cycles per second.}$$

This instrument would be unsatisfactory for frequencies below 20 cycles per second.

In addition to the error at low frequency due to the period of the arm, a spring-type instrument has an error at high frequency due to transverse vibrations of the spring. The natural frequency for transverse vibrations is,

$$n_t = \frac{1}{2l} \sqrt{\frac{T_s}{m}}.$$

T_s = tension on spring, dynes;

m = mass per unit length, grams per centimeter;

l = length of spring, centimeters.

For the spring used in the preceding computation,

$l = 3.79 \text{ cm}$; $T_s = 240 \times 980 = 235,000 \text{ dynes}$.

Cross section of wire = $\frac{\pi d^2}{4} = \frac{3.14 \times (0.014)^2}{4} = 154 \times 10^{-6} \text{ square inches}$.

Total length of wire = $N \times 2\pi R = 50 \times 6.28 \times 0.061 = 19.13 \text{ inches}$.

Volume of steel in spring = $19.13 \times 154 \times 10^{-6} = 0.00295 \text{ cubic inch}$.

Mass of spring = $0.00295 \times 127.4 = 0.376 \text{ gram}$.

Unit mass of spring = $m = \frac{0.376}{3.79} = 0.0995 \text{ gram per centimeter}$.

Natural frequency of spring = $n_t = \frac{1}{2 \times 3.79} \sqrt{\frac{235,000}{0.0995}} = 202.5 \text{ cycles per second}$.

As spring vibrations are set up not only at frequencies near the natural frequency but also at fractions of that value given by the harmonic sequence, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$. . . this spring would cause harmonics at approximately 101, 67, 50 cycles, etc., probably being very slight for other harmonic fractions below 50 cycles per second.

To reduce the value of the main frequency of the arm mass M must be increased or spring constant k decreased. The simplest way to decrease k is to increase number of turns N . This increase does not affect the safe load or the mass per unit length. To reduce the frequency of the instrument for which computations were just made to less than 1 cycle per second N may be made 10 times as large, or $N = 500$ turns and $k = 11,700 \text{ dynes per centimeter}$.

$$n = \frac{1}{6.28} \sqrt{\frac{11,700}{240}} = 0.532 \text{ cycle per second.}$$

This is a satisfactory value, but the spring is now $3.79 \times 10 = 37.9 \text{ cm}$ long; a spring of this length would produce great error from transverse spring vibrations as well as offer difficulties in construction. The natural frequency of the long spring would be,

$$n_t = \frac{202.5}{10} = 20.3 \text{ cycles per second.}$$

These considerations show that change in length of spring cannot solve the difficulty.

Change in size of spring wire, accompanied by a corresponding change in diameter of the helix to keep within the safe load value, does not improve the design. Any change that tends to lower the main frequency to a low value also causes a decrease in the frequency of the spring for transverse vibrations.

Change in M is also unsuccessful, as the larger weight requires a stronger spring, thus reducing n_t as well as n . Likewise, shortening the arm to which the spring is attached, while reducing the natural frequency, requires a stronger spring to supply the greater force. For satisfactory operation n must be made much lower than the lowest frequency for which the instrument is to be used and at the same time n_t must be made much higher than the upper frequency limit; this method of spring support is therefore unsatisfactory. Table 3 shows the results of computations for all possible changes.

TABLE 3.—Values of n and n_t for different spring dimensions

M , grams	d , inch	R , inch	N , turns	k , $\frac{\text{dynes}}{\text{centimeter}}$	n , cycles	n_t , cycles
240	0.014	0.061	50	117,000	3.52	202.5
240	.014	.061	500	11,700	.53	20.3
240	.014	.050	100	101,000	3.13	127.0
240	.020	.18	25	37,000	1.97	83.0
240	.020	.18	50	18,500	1.24	41.5
240	.028	.40	10	31,500	1.83	60.0
960	.028	.98	100	112,000	1.72	50.0

MATHEMATICAL THEORY OF GEOMETRICAL SPRING COMPENSATION

Two methods of applying the spring for geometrical compensation are shown in figure 25. There are other methods, but all can be

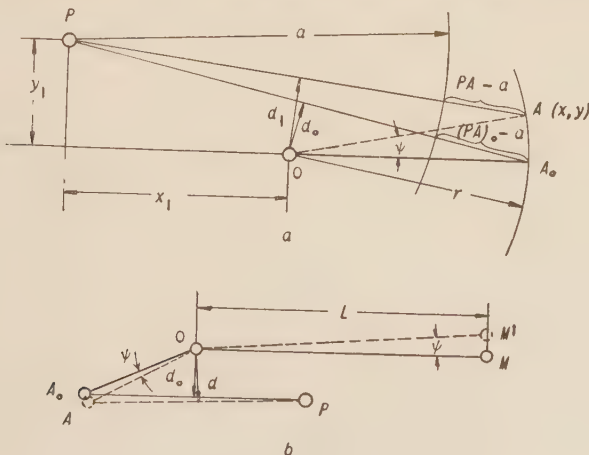


FIGURE 25.—Diagram illustrating geometrically compensated spring. Spring attached to frame at P and to arm at A .

resolved to that shown in figure 25, a . In figure 25 the axis of the hinge is taken as the origin, and the point (P) at which the spring is attached to the frame has the coordinates $-x_1$ and y_1 . The length of the spring arm is r , the coordinates of the point (A) at which the spring is attached to the spring arm are x and y , and the perpendicular

distance from 0 to the spring is d . The following relations are thus obtained:

Equation of circle about origin with radius r (circle through A),

$$x^2 + y^2 = r^2.$$

Equation of any line through P with slope S (axis of spring),

$$y - y_1 = s(x + x_1).$$

Perpendicular distance from the origin to the spring,

$$d = \frac{-sx_1 - y_1}{\sqrt{s^2 + 1}}$$

$$y = r \sin \psi \text{ and } x = r \cos \psi$$

$$s = \frac{y - y_1}{x + x_1} = \frac{r \sin \psi - y_1}{r \cos \psi + x_1}$$

$$d\sqrt{s^2 + 1} = -sx_1 - y_1$$

$$d\sqrt{\frac{(r \sin \psi - y_1)^2}{(r \cos \psi + x_1)^2} + 1} = \frac{y_1 - r \sin \psi}{x_1 + r \cos \psi} x_1 - y_1$$

$$d\sqrt{(r \sin \psi - y_1)^2 + (r \cos \psi + x_1)^2} = (y_1 - r \sin \psi) x_1 - (x_1 + r \cos \psi) y_1$$

$$d \times (PA) = -(x_1 r \sin \psi + y_1 r \cos \psi).$$

PA may be found for any value of ψ from x_1 , y_1 , and r . Then d may be found from the relation,

$$d = \frac{x_1 r \sin \psi + y_1 r \cos \psi}{PA} \text{ (neglecting the sign).}$$

For perfect compensation $(PA - a) \times d$ must be constant for small values of ψ where a is the length of the spring when it is not under tension; the force exerted by the spring is therefore proportional to $(PA - a)$. If the following values are assumed computation shows almost perfect compensation:

$$x_1 = 1.5, y_1 = 1.508, r = 1.0, \text{ and } a = 2.57.$$

TABLE 4.—*Moments of lifting force at small angles of rotation from neutral position*

$\psi, ^\circ$	PA	d	$PA - a$	$(PA - a)d$	$\psi, ^\circ$	PA	d	$PA - a$	$(PA - a)d$
+4	2.881	0.561	0.311	0.175	-2	2.938	0.496	0.368	0.182
+2	2.902	.552	.332	.183	-4	2.953	.476	.383	.182
0	2.921	.518	.351	.182					

If the value of a is increased, all other constants remaining as before, the instrument will be undercompensated and a restoring force will result; or if a is decreased the instrument will be unstable, upward swings causing an increase of pull and downward swings a decrease in pull. The effect is a negative restoring force tending to move away from the neutral point in either direction it is started. The

following table shows how the values of torque vary for an undercompensated and overcompensated instrument.

TABLE 5.—Variations in values of torque for an undercompensated and overcompensated instrument

$\psi,^\circ$	$a=2.4$, overcompensated instrument		$a=2.7$, undercompensated instrument		$\psi,^\circ$	$a=2.4$, overcompensated instrument		$a=2.7$, undercompensated instrument	
	$(PA-a)$	$(PA-a)d$	$(PA-a)$	$(PA-a)d$		$(PA-a)$	$(PA-a)d$	$(PA-a)$	$(PA-a)d$
+4	0.481	¹ 0.269	0.181	0.1015	-2	0.538	0.267	0.238	0.1180
+2	.504	.277	.202	.1115	-4	.553	.263	.253	.1204
0	.521	.269	.221	.1144					

¹ At this point the values begin to decrease for upward swings and the instrument will be undercompensated, but below -2° it is overcompensated.

COMPUTATION OF CONSTANT K FOR VERTICAL-COMPONENT SEISMOMETER WITH COMPENSATION

The force exerted by the spring is proportional to the extension $(PA-a)$. If this extension is expressed in centimeters and multiplied by spring constant k in dynes per centimeter the force in dynes exerted by the spring for any given position of the arm will be obtained:

$$\text{Force} = k(PA-a) \text{ dynes.}$$

This force multiplied by its effective radius d gives the turning moment or torque in dyne-centimeters:

$$\text{Torque} = dk(PA-a) \text{ dyne-centimeters.}$$

The lifting force at the center of gravity of the arm is found by dividing this torque by the effective length of arm L :

$$f = \frac{dk(PA-a)}{L} \text{ dynes.}$$

When the spring is undercompensated there will be one point at which this lifting force is just balanced by the downward force of gravity Mg . The arm is in stable equilibrium at this point and when displaced will tend to return to this position. If underdamped it will tend to oscillate about this point at a low frequency. To compute the frequency of this oscillation a value must be found for constant K , which in the horizontal instrument is equal to $Mg \sin \alpha$.

Let f_0 = lifting force when in the neutral position.

Let f_1 = lifting force when in some other position.

If the position for f_1 is higher than the neutral position f_1 will be less than f_0 and the arm will tend to drop. If f_1 corresponds to a position below the neutral position the force will be greater than f_0 and the arm will tend to rise. Then $f_0 = Mg$, and the restoring force is $f_0 - f_1$, which may be either plus or minus but always acts toward the neutral position and approximately at right angles to the arm, as the arm is nearly level. Constant K is the force in dynes acting in a direction parallel to the neutral position of the arm, so that the

actual restoring force corresponding to $f_0 - f_1$ is $K \sin \theta$. Therefore, K_v for the vertical instrument must be,

$$K_v = \frac{f_0 - f_1}{\sin \psi_1} \text{ dynes,}$$

in which f_0 and f_1 are found from the equation for the lifting force applied at the neutral position and some other position. In full, the equation is

$$K_v = \frac{\frac{d_0 k[(PA)_0 - a]}{L} - \frac{d_1 k[(PA)_1 - a]}{L}}{\sin \psi_1},$$

or

$$K_v = \frac{k}{L \sin \psi_1} \{ [(PA)_0 - a] d_0 - [(PA)_1 - a] d_1 \}.$$

This value of K_v may be substituted for K in all formulas given for the horizontal instrument, and the results will hold for a vertical instrument of this type.

For example, assume that the mass of the arm of the instrument for which table 4 applies is 150 grams and its effective length is 6 cm. As a horizontal instrument with an inclination of 5° it would have a constant, K , as follows: $K = Mg \sin 5^\circ = 150 \times 980 \times 0.0871 = 12,800$ dynes.

As a vertical instrument the lifting torque would be $150 \times 6 = 900$ gram-centimeters.

According to table 4 the effective radius of the spring is about 0.5 cm so that the pull of the spring would be $\frac{900}{0.5} = 1,800$ grams = 3.96 pounds.

Assume that the diameter of the spring wire is 0.028 inch. To support a pull of 3.96 pounds the mean radius of the spring is found to be,

$$R = \frac{\pi S d^3}{16 W} = \frac{3.14 \times 60,000 \times (0.028)^3}{3.96} = 0.0654 \text{ inch.}$$

The number of turns in the spring is found from a , table 5, and the diameter of the wire (undercompensated spring):

$$N = \frac{2.7}{0.028 \times 2.54} = 38.$$

The spring constant is found as follows:

$$k = \frac{Gd}{N} \left(\frac{d}{4R} \right)^3 = \frac{12 \times 10^6 \times 0.028}{38} \left(\frac{0.028}{4 \times 0.0654} \right)^3 = 7.07.$$

7.07 pounds per inch = 124,000 dynes per centimeter.

Computing the value of K_v for $+2^\circ$ and -2° from table 5:

$$\text{At } +2^\circ, K_v = \frac{124,000}{4 \times 0.0287} (0.1144 - 0.1115) = 3,130 \text{ dynes.}$$

$$\text{At } -2^\circ, K_v = \frac{124,000}{4 \times (-0.0287)} (0.1144 - 0.1180) = 3,890 \text{ dynes.}$$

The mean of these two values is 3,510, only about one fourth of the value for the same arm in a horizontal instrument with an inclination of 5° .

Using this value to compute the natural frequency with no damping,

$$n = \frac{1}{2\pi} \sqrt{\frac{K_v}{ML}} = \frac{1}{6.28} \sqrt{\frac{3,510}{150 \times 6}} = 0.315 \text{ cycle per second.}$$

This value agrees fairly well with the value found by experiment on an instrument of similar design with approximately these constants.

LIST OF SYMBOLS

- $\pi = 3.1416$.
 $e = 2.7183$, base of Napierian logarithms.
 $p = \text{operator, } \frac{d}{dt}$.
 $j = \sqrt{-1}$.
 $g = 980 \text{ cm per second}^2$, acceleration of gravity.
 $A = \text{constant of integration}$.
 $B = \text{constant of integration}$.
 $C = \frac{ZX_1}{LM\omega^2 + K}$, amplitude factor, centimeters.
 $F = \text{force applied to spring, pounds}$.
 $F_H = \text{force due to damping action, dynes}$.
 $F_M = \text{force due to inertia of arm, dynes}$.
 $F_R = \text{force due to restoring torque, dynes}$.
 $G = \text{modulus of torsional elasticity, pounds per square inch}$.
 $H = \text{damping constant, gram-centimeters per second}$.
 $K = Mg \sin \alpha$, force causing restoring torque, dynes.
 $K_v = \text{constant for computing restoring force on arm of compensated spring-type vertical-component seismometer, dynes}$.
 $L = \text{effective length of pendulum to center of oscillation, centimeters}$.
 $M = \text{mass of seismometer arm, grams}$.
 $N = \text{number of turns in helical spring}$.
 $PA = \text{instantaneous value of length of spring, centimeters}$.
 $R = \text{mean radius of spring helix, inches}$.
 $S = \text{elastic constant for spring metal, pounds per square inch}$.
 $T = \text{period in seconds}$.
 $T_s = \text{tension in spring, dynes}$.
 $W = \text{safe load on helical spring, pounds}$.
 $X_1 = \text{maximum value of sine wave } x_1, \text{ centimeters}$.
 $X_2 = \text{maximum value of sine wave } x_2, \text{ centimeters}$.
 $X_3 = \text{maximum value of sine wave } x_3, \text{ centimeters}$.
 $\hat{X}_1 = \text{vector generating sine wave } x_1$.
 $\hat{X}_2 = \text{vector generating sine wave } x_2$.
 $\hat{X}_3 = \text{vector generating sine wave } x_3$.
 $Z = \sqrt{(H\omega)^2 + K^2}$, constant of force, dynes.
 $a = \frac{H}{2LM}$, factor of attenuation, reciprocal seconds.
 $\bar{a} = \text{normal length of spring with no tension, centimeters}$.
 $\bar{d} = \text{instantaneous value of distance from pivot to axis of spring, centimeters}$.
 $d_o = \text{distance from pivot to axis of spring for neutral position, centimeters}$.
 $d_1 = \text{distance from pivot to axis of spring when arm is rotated through angle } \psi_1, \text{ centimeters}$.
 $f = \text{lifting force at any position, dynes}$.
 $f_o = \text{lifting force at neutral position, dynes}$.
 $f_1 = \text{lifting force at position rotated through angle } \psi_1, \text{ dynes}$.
 $k = \text{spring constant, dynes per centimeter}$.
 $l = \text{length of spring, centimeters}$.
 $m = \text{mass of spring per unit length, grams per centimeter}$.
 $n = \text{natural frequency of arm, cycles per second}$.
 $n_t = \text{natural frequency of spring for transverse vibrations, cycles per second}$.
 $r = \text{distance from pivot to point at which spring is attached to arm, centimeters}$.
 $s = \text{slope of line through } P \text{ in figure 25}$.
 $t = \text{time, seconds}$.
 $x = \text{extension of spring, centimeters}$.

- x_1 = instantaneous displacement of seismometer, centimeters.
 x_2 = instantaneous displacement of center of oscillation of arm, centimeters.
 $x_3 = x_1 + x_2$, centimeters.
 $x_2' = \frac{dx_2}{dt}$, velocity of motion at center of oscillation, centimeters per second.
 $x_2'' = \frac{d^2x_2}{dt^2}$, acceleration of motion at center of oscillation, centimeters per second².
 $(x_2)_0$ = initial value of x_2 , centimeters.
 $(\dot{x}_2)_0$ = initial value of velocity $\frac{dx_2}{dt}$, centimeters per second.
 α = angle of inclination from horizontal, degrees.
 $\beta = \tan^{-1} \frac{H\omega}{K}$, degrees.
 $\gamma = \tan^{-1} \frac{2\omega\sqrt{KLM}}{K - \omega^2 LM}$, degrees.
 $\delta = \beta - \gamma$, phase angle between curves of x_1 and x_2 , degrees.
 ν = frequency in cycles per second.
 ϕ = phase angle between curves x_1 and x_3 , degrees.
 ψ = instantaneous value of angular displacement in vertical-component seismometer, degrees.
 ψ_1 = value of angular displacement at a certain position, degrees.
 θ = angle of rotation of arm in seismometer, degrees.
 $\omega = 2\pi$ times the frequency, angular velocity, radians per second.

SYMBOLS USED BY TIMOSHENKO

- A = amplitude of recorded curve.
 b = constant for a seismometer.
 $p = 2\pi$ times natural frequency of seismometer, radians per second.
 $m = 2\pi$ times frequency of applied vibration, radians per second.

SYMBOLS USED BY ANDERSON AND WOOD

- $\mathcal{A}$ = applied angular acceleration, maximum value, radians per second²
 $M = 4\pi^2 \left(\frac{1}{T_o^2} + \frac{1}{T_e^2} \right)$.
 T_o = natural period of seismometer, seconds.
 T_e = period of applied vibration, seconds.



V I T A

George Allison Irland was born in Lewisburg, Pennsylvania, in 1894. He graduated from Bucknell University, receiving the degree of Bachelor of Science in Electrical Engineering, in 1915; was employed by the Bethlehem Steel Company, in the electrical department of their Maryland Plant, at Sparrow's Point, Maryland, for two years; served one year in the United States Signal Corps; returned to the Bethlehem Steel Company in 1919, and remained in their employ for one and a half years. In 1920 he was appointed instructor of electrical engineering at Bucknell University, and received the degree of Electrical Engineer from that institution in 1922. He spent the academic year of 1924-25 at The Johns Hopkins University receiving the degree of Master of Electrical Engineering in June 1925. He then returned to Bucknell University as Assistant Professor of Electrical Engineering, which position he now holds.

